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Original article

Invariant distribution of stochastic Gompertz equation under regime switching ☆

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Abstract

This paper is concerned with the asymptotic behaviors of a stochastic Gompertz model in random environments from the view of Itô stochastic differential equations with Markovian switching. Based upon the deterministic Gompertz model, we establish the corresponding stochastic model which is described as a stochastic Gompertz models with Markovian switching. We show that this model is asymptotically stable in distribution and that it displays an invariant probability distribution under certain conditions. Most importantly, we simulate the trajectories and the limits probability distribution of the solution with the method of Monte Carlo stochastic simulation. The simulation results illustrate that our conclusions are correct, and moreover the results reflect the statistical properties of the stochastic model.

MSC: 60H10; 92B05; 81T80

Keywords: Gompertz model; Markov chains; Stationary distribution; Stochastic simulation

1. Introduction: background and research aims

Mathematical models have come to play an important role in describing the changes and growth of a population. The classical deterministic, autonomous logistic equation, $dX(t) = rX(t)(1 - X(t)/K)dt$ is often used to describe the growth of a single species. For this model there is a stable equilibrium point of the equation, and many authors have obtained a variety of interesting results with this model. However, in the real world the growth of the population for many species does not obey the logistic equation very perfectly. Gompertz growth model is considered to be better fitted to the reality of certain types of tumor growth. For example, the Gompertz growth model provides an excellent fit to the empirical growth curves of avascular tumors and vascular tumors in their early stages (see [1,23]). The Gompertz model has been almost universally used to describe the growth of microorganisms (see [8]) and the innovation diffusion such as digital cellular telephones [5]. The Gompertz model is another important type of mathematical model and, for example, the growth of the industrial production, the life cycle of a product, and population growth over a certain period all comply with this model.

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Gompertz [6] established the following equation to describe the growth of a species

$$dX(t) = rX(t)\ln\left(\frac{K}{X(t)}\right)dt \tag{1.1}$$

where $X(t)$ represents the density of the population at time t, r is the intrinsic growth rate and K is the carrying capacity. Eq. (1.1) was proposed as a model to express the law of human mortality and it can also be used for population estimation. Parfitt and Fyhrie [30] developed a methodology for statistical estimation of the parameters for the deterministic Gompertz growth function using graphical methods. The authors of [7,22,32] all studied population models described by the deterministic Gompertz equation. However, in the real world, the growth of a population is often subject to various forms of environmental noise which may affect the essential property of the system. Hence it becomes necessary to consider noise of different forms when establishing mathematical models to describe the growth law of a population. Therefore the deterministic growth models have been extended to stochastic versions by many researchers and subsequently probabilistic and stochastic methodology have been introduced to deal with the stochastic models. Noting some important advantages of stochastic models based on diffusions, Moummou et al. [27] used the maximum likelihood method estimated the parameters of interest in the drift coefficient of a stochastic Gompertz model with logarithmic therapeutic functions and obtained an explicit expression for the parameter of the tumor growth deceleration factor. Gutiérrez et al. [11,12,14] applied a similar methodology to the modeling of tumor growth. Nafidi [28] defined a stochastic Gompertz diffusion process from the perspective of the Kolmogorov equations. Later, Gutiérrez et al. [10] studied the growth in the price of new housing in Spain, the consumption of electricity in Morocco, and studied the stock of motor vehicles in Spain [13].

However, as far as the author knows, few results can available about the properties of stochastic Gompertz model with Markovian switching. In recent years, many researchers have paid significant attention to the study of stochastic models based on stochastic differential equations. Mao et al. [24–26] investigated many properties of stochastic differential equations in general, including different forms of stochastic stability and many useful methods to study the stochastic models. Moreover, there are also many publications, such as [3,9,18,29], about the theory of stochastic differential equation.

The remaining part of this paper is organized as follows. In Section 2, we describe the stochastic Gompertz model with Markovian switching in detail. In Section 3, we obtain the main results of this paper: that is the stochastic Gompertz equation with Markovian switching is asymptotically stable in distribution. In order to clarify the limits probability distribution of an stochastic model, and to support the theoretical result, we provide computer simulation results for the sample trajectories of the solution and for the invariant distribution of the stochastic Gompertz model with Markovian switching in Section 4. Finally, we discuss the results of our paper and make suggestions about future research directions in Section 5.

2. Description of the model and research basis

Gompertz model and logistic model are two mathematical growth models associated with bounded curves. Though the two models have a similar growth pattern, their difference is that, the Gompertz curve reaches an inflection point where the curve changes from concave to convex in the first part of the growth cycle whereas the logistic curve reaches this point in the middle. The logistic model is suitable to fit data showing a bounded growth for which the bound does not depend on the initial value while Gompertz curve has undergone several changes and has been stated in diverse forms to facilitate its study and bound of Gompertz curve depends on the initial value. Gutiérrez-Jáimez et al. [15] introduced a new Gompertz-type diffusion process, did some comprehensive study to this model. In consequence, it is very necessary to introduce Gompertz-type diffusion process which is described by the Itô stochastic Gompertz equation with Markovian switching.

Let us first introduce colored noise into Eq. (1.1), say telegraph noise which is often illustrated as a switching between two or more regimes of the states. The switching is memoryless, and the waiting time for the next switch obeys an exponential distribution. Hence we can model this form of regime switching by a finite-state Markov Chain as in [21]. Assume that the system obeys

$$dX(t) = r(i)X(t)\ln\left(\frac{K(i)}{X(t)}\right)dt$$