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t-Copula generation for control variates

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Abstract

The standard method for generating multi-t vectors is simple and convenient but it has the disadvantage that the generated multi-normal and multi-t vectors are not similar. For t-copula models this destroys much of the variance reduction when using the result of the multi-normal model as external control variate. Therefore we develop a new generation method for multi-t vectors. It is based on the polar method and numerical inversion, and generates multi-normal and multi-t vectors that are very similar. Numerical experiments with simple functions of the weighted sum of t-copula vectors and with pricing European basket options with a t-copula model confirm that the obtained variance reduction factors of the new method are high; 2–100 times higher than when using the standard generation method.

Keywords: Monte Carlo simulation; t-Copula; Control variate; Polar method

1. Introduction

The practical importance of copula for modeling dependence is well accepted today. For example in financial applications the t-copula has become a standard tool, especially for risk quantification. See, e.g. [2,6,9,10,13]. Unfortunately the use of the t-copula implies that even simple problems like the calculation of the variance or of tail loss probabilities of weighted sums do not have closed-form solutions. This makes the development of efficient Monte Carlo methods for t-copula a practical important task.

Many of the multivariate distributions relevant in practice can be approximated by the multi-normal distribution. If the selected marginals are not too different from normal and the degrees of freedom of the t-copula not too small it is also true that t-copula models can be approximated by the multi-normal distribution. Many simple problems (like the two mentioned above) have closed form solutions for the multi-normal distribution. It is therefore a natural idea to try to achieve variance reduction by using the approximating multi-normal model as external control variate (CV). But it turns out that this idea leads only to moderate variance reduction factors (for most examples between 2 and 10). This is even true when the distribution of the used t-copula is very similar to the multi-normal distribution. How is that possible? The standard approach for generating vectors from the multi-t distribution, which is the most common

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found in the literature, requires a normal vector and an additional independent random variate. Thus the correlations between the entries of the normal vector and the t-vector are reduced, which in turn reduces the variance reduction. To overcome this problem we present the polar method to generate vectors from the multi-t distribution.

Based on the polar method it is possible to develop a new algorithm that generates very similar multi-normal and multi-t vectors, as we explain in detail in Section 2. Section 3 demonstrates the use of the new generation method for external CV algorithms for t-copula models, whereas Section 4 reports our numerical experience for several basic examples. The results for pricing a European basket option with a t-copula model are presented in Section 5, while Section 6 contains our conclusions.

2. The generation of multi-t vectors

2.1. The standard method

It is well known that a t-variate T can be generated as the ratio of a standard normal variate Z over the square root of a chi-square variate X, which is independent of Z, divided over its degrees of freedom ν. That formula

$$T = \frac{Z}{\sqrt{X/\nu}} \tag{1}$$

remains valid if we interpret Z as a vector of d i.i.d. standard normal variates and divide all its components by the single chi-square random variate X with ν degrees of freedom. The resulting vector T follows the multi-t distribution with parameter matrix Σ equal to the identity matrix. This method is simple and fast and certainly a good method to generate vectors from the multi-t distribution. It is also easy to see that, as mentioned in the introduction, the vectors Z and T are not very similar, unless ν is very large. Therefore the standard generation method does not lead to large variance reduction for CV applications.

Remark: To obtain a different Σ it is necessary to transform T by the Cholesky-factor L of Σ which satisfies $LL' = \Sigma$. As this linear transform is no problem in practice we discuss for the moment only the case that Σ is the identity matrix.

2.2. The polar method

As the multi-normal distribution is radially symmetric it is clear from (1) that the multi-t distribution is radially symmetric as well. It is therefore possible to generate the multi-t vector T of length d using the polar method. First a vector U_S distributed uniformly on the unit-sphere S^{d-1} has to be generated (easiest way: generate a vector of i.i.d. $N(0, 1)$ variables and normalize it to length one [5]). Then the random variate R_T (the length of the radius) has to be generated independently and we obtain the vector T using

$$T = R_T\, U_S. \tag{2}$$

Using the result of Theorem 4.3 of [5, p. 229] it is not difficult to calculate, that for the multi-t distribution the radius variate R_T has density:

$$f_{R_T}(r) = \frac{\Gamma((\nu+d)/2)}{\Gamma(\nu/2)\,\Gamma(1+d/2)\nu^{d/2}}\, d\, r^{d-1}(1+r^2/\nu)^{-((\nu+d)/2)}\, 1_{[0,+\infty)}(r) \tag{3}$$

where $1_{[0,+\infty)}(r)$ is the indicator function of $[0, +\infty)$.

For dimension $d = 2$ integration and inversion of (3) leads to the inverse cumulative distribution function (CDF):

$$F_{R_T}^{-1}(u) = \sqrt{\nu\,\left((1-u)^{-2/\nu} - 1\right)}, \tag{4}$$

which is used in [1] to generate a pair of the multi-t distribution. Actually [1] suggests using the polar method in dimension two as a fast and simple method for generating a one-dimensional t-variate.

For dimension $d > 2$ and ν an integer, it is still possible to find a closed-form solution for the CDF of R by integrating (3) but inversion of the CDF is no longer possible in closed form. It is therefore necessary to use either one of the automatic algorithms for T-concave distributions described in [8] or numerical inversion. Our new numeric inversion algorithm NINGL (see [3,4]) requires only the density and is thus easily used to generate the random variate R_T. We present all details to generate multi-t vectors using the polar method as Algorithm 1.