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Bivariate orthogonal polynomials on triangular domains

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Abstract

In the paper [A. Rababah, M. Alqudah, Jacobi-weighted orthogonal polynomials on triangular domains, J. Appl. Math. 3 (2005) 205–217.], Jacobi-weighted orthogonal polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u, v, w)$, $\alpha, \beta, \gamma > -1$ on the triangular domain T for values of $\alpha, \beta, \gamma > -1$ in the plane $\alpha + \beta + \gamma = 0$ are constructed. In this paper, the results are generalized to every point in the space $\forall \alpha, \beta, \gamma > -1$.

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Keywords: Bivariate orthogonal polynomials; Bernstein polynomials; Jacobi polynomials; Triangular domains

1. Introduction and definitions

Jacobi-weighted orthogonal polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u, v, w)$ on the triangular domain T for every point in the half space $\alpha, \beta, \gamma > -1$ are constructed. These polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u, v, w) \in \mathcal{L}_n$, $r = 0, 1, \ldots, n$, and for $r \neq s$ they satisfy $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u, v, w) \perp \mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}(u, v, w)$. Consequently, for $n \geq 1$, the bivariate polynomials $\{P_{n,r}^{(\alpha,\beta,\gamma)}(u, v, w)\}_{r=0}^{n}$ form orthogonal system over the triangular domain T. These results are generalizations of the results in Refs. [4,9,10], see also [1,2,11,13]. We end this section by giving some definitions: The univariate Bernstein polynomials $b_i^n(u)$, $u \in [0, 1]$, $i = 0, 1, \ldots, n$ are defined by

$$b_i^n(u) = b_i^n(u, 1-u) = \begin{cases} \dfrac{n!}{i!(n-i)!} u^i (1-u)^{n-i}, & i = 0, 1, \ldots, n \\ 0, & \text{else} \end{cases} \tag{1}$$

The univariate Jacobi polynomials $P_n^{(\alpha,\beta)}(u)$ of degree n are the orthogonal polynomials on [0, 1] with respect to the weight function

$$w(u) = (2 - 2u)^{\alpha} (2u)^{\beta}, \quad \alpha, \beta > -1. \tag{2}$$

Let $T = \Delta \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3$ be a reference triangle in the plane with vertices $\mathbf{p}_k = (x_k, y_k)$, $k = 1, 2, 3$. Then every point $\mathbf{p}$ inside the triangle T is uniquely written in the form $\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$, where (u, v, w) are the barycentric coordinates

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with $u + v + w = 1, \quad u, v, w \geq 0$. The bivariate Bernstein polynomials of degree n on T are defined by the formula

$$b^n_\alpha(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k, \quad |\alpha| = i + j + k = n. \tag{3}$$

Π_n denotes the space of all polynomials of total degree n over T.

$$\mathcal{L}_n = \{p \in \Pi_n : \quad p \perp \Pi_{n-1}\}.$$

The inner product of the polynomials $P(u, v, w)$ and $Q(u, v, w)$ over T with respect to the Jacobi weight function $W^{(\alpha,\beta,\gamma)}(u, v, w) = u^\alpha v^\beta (1 - w)^\gamma$, $\alpha, \beta, \gamma > -1$ is given by

$$\langle P, Q\rangle = \frac{1}{\Delta} \int\int_T PQW^{(\alpha,\beta,\gamma)}(u, v, w)\, \mathrm{d}A. \tag{4}$$

For more, see [3,6–8,12].

2. Bivariate orthogonal polynomials

Let $\sigma = \alpha + \beta + \gamma$, then the following lemmas will be used in the proof of Theorem 3.

Lemma 1. *The following identity holds*

$$S = \sum_{j=0}^{n-r} (-1)^j \frac{\binom{n+r+\sigma+1}{j}\binom{n-r}{j}}{\binom{n+r+i+\sigma+1}{j}} = \frac{\binom{i}{n-r}}{\binom{n+r+i+\sigma+1}{n-r}}$$

Proof. Using Eq. (5.21) in Ref. [5], and negating the binomial term in the numerator, we get

$$\frac{\binom{n-r}{j}}{\binom{n+r+i+\sigma+1}{j}} = (-1)^{n-r-j} \frac{\binom{-2r-i-\sigma-2}{n-r-j}}{\binom{n+r+i+\sigma+1}{n-r}}.$$

Substituting these simplifications in the summation, we get

$$S = \frac{(-1)^{n-r}}{\binom{n+r+i+\sigma+1}{n-r}} \sum_{j=0}^{n-r} \binom{n+r+\sigma+1}{j}\binom{-2r-i-\sigma-2}{n-r-j}.$$

Using Eq. (5.22) in Ref. [5], we have

$$S = \frac{(-1)^{n-r}\binom{n-r-i-1}{n-r}}{\binom{n+r+i+\sigma+1}{n-r}},$$

and by negating the numerator, the identity holds. □

We define the polynomials $q_{n,r,\sigma}(w)$ of degree $n - r$ by

$$q_{n,r,\sigma}(w) = \sum_{j=0}^{n-r} (-1)^j \binom{n+r+\sigma+1}{j} b^{n-r}_j(w), \quad r = 0, 1, \ldots, n. \tag{5}$$