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Mathematics and Computers in Simulation 81 (2010) 560–567

MATHEMATICS AND COMPUTERS IN SIMULATION

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Quasi-Monte Carlo methods for Markov chains with continuous multi-dimensional state space

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Received 4 September 2007; received in revised form 19 October 2009; accepted 29 July 2010
Available online 6 August 2010

Abstract

We describe a quasi-Monte Carlo method for the simulation of discrete time Markov chains with continuous multi-dimensional state space. The method simulates copies of the chain in parallel. At each step the copies are reordered according to their successive coordinates. We prove the convergence of the method when the number of copies increases. We illustrate the method with numerical examples where the simulation accuracy is improved by large factors compared with Monte Carlo simulation.

Keywords: Markov chain; Discrepancy; Quasi-Monte Carlo method; Simulation

1. Introduction

Many real-life systems can be modeled using Markov chains. Fields of application are queueing theory, telecommunications, option pricing, etc. In most interesting situations, analytic formulas are not available and the state space of the chain is so large that classical numerical methods would require a considerable computational time and huge memory capacity. So Monte Carlo (MC) simulation becomes the standard way of estimating performance measures for these systems. A drawback of MC methods is their slow convergence. One approach to improve the accuracy of the method is to change the random numbers used. Quasi-Monte Carlo (QMC) methods use quasi-random numbers instead of pseudo-random numbers. Pseudo-random numbers aim to simulate a sequence of independent and identically distributed (i.i.d.) random variables with a given distribution (we only consider the uniform distribution). In the example of MC integration, it is not so much the randomness of the samples that is relevant, but rather that the samples should be spread in a uniform manner over the integration domain. Quasi-random numbers are sample points for which the empirical distribution is close to the uniform distribution; unlike for random sampling, quasi-random points are not required to be independent and may be completely deterministic.

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0378-4754/$36.00
doi:10.1016/j.matcom.2010.07.027

The efficiency of a QMC method depends on the quality of the quasi-random points that are used. Broadly speaking, these points should form a low-discrepancy point set. We recall from [12] some basic notations and concepts. We first denote $\mathcal{I}:=[0, 1)$. Let $s \geq 1$ be a fixed dimension and denote by λ_s the s-dimensional Lebesgue measure. For a set $U = \{\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}\}$ of points in the s-dimensional unit cube $\mathcal{I}^s$ and for a Borel set $B \subset \mathcal{I}^s$ we define the *local discrepancy* by

$$D(B, U):=\frac{1}{N}\sum_{0\leq k<N} 1_B(\mathbf{u}_k) - \lambda_s(B), \tag{1}$$

where 1_B denotes the indicator function of B. The *discrepancy* of U is defined by $D(U) := \sup_Q |D(Q, U)|$, the supremum being taken over all subintervals $Q \subset \mathcal{I}^s$. The *star discrepancy* of U is $D^\star(U):=\sup_{Q^\star}|D(Q^\star, U)|$, where $Q^\star$ runs through all subintervals of $\mathcal{I}^s$ of the form $\prod_{i=1}^{s}[0, a_i)$. A *low-discrepancy* point set in $\mathcal{I}^s$ is a set of N points for which the discrepancy is of size $\mathcal{O}((\log N)^{s-1}/N)$, which is the minimum size possible. The most powerful current methods of constructing low-discrepancy point sets are based on the theory of (t, m, s)-nets. For an integer $b \geq 2$, an *elementary interval in base b* is an interval of the form $\prod_{i=1}^{s}[a_i b^{-d_i}, (a_i + 1)b^{-d_i})$, with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$. If $0 \leq t \leq m$ are integers, a (t, m, s)-*net in base b* is a point set U consisting of b^m points in $\mathcal{I}^s$ such that $D(Q, U) = 0$ for every elementary interval Q in base b with measure b^{t-m}. If $b \geq 2$ and $t \geq 0$ are integers, a sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots$ of points in $\mathcal{I}^s$ is a (t, s)-*sequence in base b* if, for all integers $j \geq 0$ and $m > t$, the points $\mathbf{u}_\ell$ with $jb^m \leq \ell < (j+1)b^m$ form a (t, m, s)-net in base b.

In the example of numerical integration, the QMC method achieves a significantly higher accuracy than the MC method, with the same computational effort. It may be hoped that the improvement obtained by using quasi-random points in place of random samples can also be attained in problems of numerical analysis that can be reduced to numerical integration. QMC simulations can outperform MC simulations in some applications: we refer to the IMACS Seminars on Monte Carlo Methods [1,2,4,13].

In previous communications, we first proposed QMC schemes to simulate Markov chains with a *discrete* state space, either one-dimensional [7,8] or multi-dimensional [3]. We next applied the method to one-dimensional continuous state spaces [10,11]. In the present work, we extend the QMC algorithm to Markov chains with *continuous multi-dimensional* state spaces.

2. The method

Our setting is an homogeneous Markov chain $\{X_j,\ j \in \mathbb{N}\}$ whose state space E is a subspace of $\mathbb{R}^s$ for some $s \in \mathbb{N}^*$. The distribution P_0 of X_0 is known, and we assume that the chain evolves according to the stochastic recurrence:

$$X_{j+1} = \varphi_{j+1}(X_j, U_{j+1}), \quad j \geq 0, \tag{2}$$

where $\{U_j,\ j \geq 1\}$ is a sequence of i.i.d. uniform random variables over $\mathcal{I}^d$ for some $d \in \mathbb{N}^*$, and $\varphi_{j+1} : E \times \mathcal{I}^d \to E$ is a measurable map for each j.

To approximate the Markov chain by ordinary MC, we proceed as follows. Given a large integer N, we draw N samples $\mathbf{x}_k^0$, $0 \leq k < N$ from the initial distribution P_0. Then for each k, we generate a sample path of the chain via

$$\mathbf{x}_k^{j+1} = \varphi_{j+1}(\mathbf{x}_k^j, \mathbf{u}_k^{j+1}), \quad j \geq 0, \tag{3}$$

where $\mathbf{u}_k^1, \mathbf{u}_k^2, \ldots$ are pseudo-random numbers which simulate independent and uniformly distributed random variables over $\mathcal{I}^d$. In order to construct a QMC algorithm for the approximation of the Markov chain, we reduce the simulation to numerical integration.

We denote by $\mathcal{M}^+$ the set of all nonnegative measurable functions on E. If P_j denotes the distribution of X_j, then

$$\forall f \in \mathcal{M}^+ \quad \int_E f\mathrm{d}P_{j+1} = \int_{\mathcal{I}^d}\int_E f \circ \varphi_{j+1}(\mathbf{x}, \mathbf{u})\, \mathrm{d}P_j(\mathbf{x})\, \mathrm{d}\mathbf{u}. \tag{4}$$

For $\mathbf{x} \in E$, let us write $\delta_\mathbf{x}$ for the unit mass at $\mathbf{x}$. We are looking for an approximation of P_j of the form

$$\hat{P}_j:=\frac{1}{N}\sum_{0\leq k<N}\delta_{\mathbf{x}_k^j}, \tag{5}$$