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The use of Sherman–Morrison formula in the solution of Fredholm integral equation of second kind

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Abstract

We consider a constructive method for the solution of Fredholm integral equations of second kind. This method is based on a simple generalization of the well-known Sherman–Morrison formula to the infinite dimensional case. In particular, this method constructs a sequence of functions, that converges to the exact solution of the integral equation under consideration. A formal proof of this convergence result is provided for the case of Fredholm integral equations with $\mathcal{L}^2$ integral kernel. Finally, a boundary value problem for the Laplace equation is considered as an example of the application of the proposed method.

Keywords: Fredholm integral equation; Iterative method; Sherman–Morrison formula

1. Introduction

Integral equations provide a natural mathematical formulation for a variety of applied problems, such as for example diffraction problems, elasticity problems, and electrostatic problems, see [6] for a detailed description of such applications. Moreover, they can be used to obtain an equivalent formulation of a large class of initial and boundary value problems for partial differential equations, see, for example, the potential theory for the Laplace operator [7, chapter 6], and the Green function theory for ordinary differential equations [12, p. 274]. So, integral equations are fundamental tools in applied mathematics and their study can be profitable used in several application fields.

We consider the following Fredholm integral equation of second kind,

$$u(s) - \lambda \int_a^b K(s,t)u(t)dt = f(s), \quad a \le s \le b \tag{1}$$

where $K : [a,b] \times [a,b] \to \mathbb{C}$ is an $\mathcal{L}^2$ integral kernel, $f : [a,b] \to \mathbb{C}$ is the $\mathcal{L}^2$ right-hand side function, $\lambda \in \mathbb{C}$, and $u : [a,b] \to \mathbb{C}$ is the $\mathcal{L}^2$ solution of the integral equation.

We assume that

$$K(s,t) = \lim_{N\to\infty} K_N(s,t), \ s,t \in [a,b] \tag{2}$$

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and, for each $N \in \mathbb{N}$, K_N is a finite rank kernel. The solution of integral Eq. (1) can be performed by a standard tool, i.e. the Fredholm theory, see [11] for a wide description. Moreover, numerical methods such as collocation, Galerkin, Petrov Galerkin and Wavelet Petrov Galerkin methods are used for the approximate solution of (1), see [1,7,2,8] for their analysis.

Note that, from Fredholm theory, the solution of Eq. (1) is based on kernels K_N appearing in (2). In fact, when K is substituted by K_N, the solution u_N of integral Eq. (1) can be obtained by solving a linear system, and u_N converges to the solution of (1) as $N \to \infty$.

In [4] we present a new recursive method for the solution of integral Eq. (1), for the real case. This method, based on a generalization of the well known Sherman–Morrison formula, provides an explicit construction of a function sequence, that converges to the solution of the Fredholm integral equation under consideration. Really, in [4] an evidence of such a convergence result has been given only by several examples. In this paper we give a generalization of this method to the complex case and the rigorous proof of the above mentioned convergence result.

The proposed method is similar to the direct approach obtained from the Fredholm theory, in fact the approximate solution u_N is computed by solving the Fredholm integral equation with kernel K replaced by a finite rank kernel K_N. The main difference between this two methods is that in the Fredholm theory u_N is computed by solving a finite linear system, in the proposed method u_N is computed recursively from the previous approximation u_{N-1}. Note that this is an important feature, for example, when we have to compute the solution of (1) with a prescribed precision.

Finally, the recursive method is applied to the solution of a boundary value problem for the Laplace equation. Note that this is a very classical application of the integral equations, in fact, historically the main goal for the development of integral equations theory was the applications to the boundary valued problems for the Laplace equation. In this example we show that the sequence constructed by the proposed method converges to the solution of the boundary value problem under consideration.

In Section 2 we recall some fundamental results of the Fredholm theory. In Section 3 we present the recursive method and we prove the convergence result for the sequence constructed by this method. In Section 4 we describe the application of the proposed method to the solution of a boundary value problem for the Laplace equation, and the corresponding numerical implementation. In Section 5 we give some final considerations about the proposed method.

2. The Fredholm theory

We recall, for the convenience of the reader, some fundamental results of the Fredholm theory, that are explained with more details in [11]. We give some notations. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers. Let $x \in \mathbb{C}$ we denote with $|x|$ the module of x, and with $\bar{x}$ the conjugate of x. Let $n, m \in \mathbb{N}$, $\mathbb{R}^n$, $\mathbb{C}^n$ are the real and complex Euclidean spaces, respectively; $\mathbb{C}^{n\times m}$ is the set of complex matrices having n rows and m columns. Let $\underline{x} \in \mathbb{C}^n$ a complex column vector, we denote with $\underline{x}^t$ the transpose of $\underline{x}$, with $\underline{x}^h$ the conjugate transposed of $\underline{x}$, and with $||\underline{x}||$ the Euclidean norm of $\mathbb{C}^n$. Note that the same notation is used for $\underline{x} \in \mathbb{R}^n$, but in this case $\underline{x}^h = \underline{x}^t$. We denote with $A = (a_{ij}) \in \mathbb{C}^{n\times m}$ the matrix whose entries are a_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and for $n = m$, we denote with $|A|$ the determinant of A. We denote with $I_n = (\delta_{ij}) \in \mathbb{C}^{n\times n}$ the identity matrix, where, for $i, j = 1, 2, \ldots, n$, $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$. Let $J \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$, we denote with $\mathcal{L}^2(J)$ the space of measurable functions $f : J \to \mathbb{C}$, that are square integrables. Let $g, u \in \mathcal{L}^2(J)$, we denote with $\langle g, u\rangle = \int_J g(t)\overline{u(t)}dt$, the usual hermitian inner product in $\mathcal{L}^2(J)$ and with $||u|| = (\langle u, u\rangle)^{(1/2)}$ the $\mathcal{L}^2(J)$ norm of u. Note that, we abuse of this notation also for Euclidean norms of $\mathbb{R}^n$ and $\mathbb{C}^n$. In the following we denote with $J = [a, b] \subset \mathbb{R}$ a bounded interval and we suppose $\lambda \neq 0$ in (1), in fact the case $\lambda = 0$ is trivial.

Definition 2.1. An $\mathcal{L}^2$ kernel is a function $K : J \times J \to \mathbb{C}$ such that

(1) $K \in \mathcal{L}^2(J \times J)$;
(2) for each $s \in J$, $K(s, \cdot) \in \mathcal{L}^2(J)$, where $K(s, \cdot) : J \to \mathbb{R}$ is the measurable function of one variable, obtained by fixing the first variable;
(3) for each $t \in J$, $K(\cdot, t) \in \mathcal{L}^2(J)$, where $K(\cdot, t) : J \to \mathbb{R}$ is the measurable function of one variable, obtained by fixing the second variable.