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Original article

A mollification regularization method for stable analytic continuation☆

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Abstract

In this paper, we consider an analytic continuation problem on a strip domain with the data given approximately only on the real axis. The Gauss mollification method is proposed to solve this problem. An *a priori* error estimate between the exact solution and its regularized approximation is obtained. Moreover, we also propose a new *a posteriori* parameter choice rule and get a good error estimate. Several numerical examples are provided, which show the method works effectively.

AMS Classification: 30B40; 65J20

Keywords: Analytic continuation; Ill-posed problems; Mollification method; Error estimate; *A posteriori*

1. Introduction

The analytic continuation is a classical problem in complex analysis, which is frequently encountered in many practical applications [6,15,19,22], while the stable numerical analytic continuation is a rather difficult problem. In general, this problem is severely ill-posed. Some theoretical and numerical studies have been devoted to the problem. The earlier works mainly focused on the results of the conditional stability [12,15]. However, it seems that there are few applications of modern theory of regularization methods which have been developed intensively in the last few decades. A simple computer algorithm was given in [6] that is based on the fast Fourier transform. In [2,24], a hypergeometric summation method was used to reconstruct some analytic functions from exponentially spaced samples, where the approximation errors and stability estimates were obtained.

In this paper, we consider the following problem of analytic continuation. Let function $f(z) = f(x + iy)$ be analytic on a strip domain D of the complex plane defined by

$$D := \{z = x + iy \in \mathbb{C} | x \in \mathbb{R}, \quad |y| < y_0, \quad y_0 \text{ is a positive constant}\}, \tag{1.1}$$

where i is the imaginary unit. The data is only given on the real axis, i.e., $f(z)|_{y=0} = f(x)$ is known approximately and we will extend f analytically from this data to the whole domain D.

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One can refer to some literatures for its important practical applications, e.g., medical imaging [5] and integral transformation [1]. It is well known that this problem is ill-posed [7–9] and some regularization methods are needed. In the earlier paper [9], D.N. Hào et al. used the mollification method with Dirichlet kernel for the case of $f(\cdot+iy)\in L^2(\mathbb{R})$ and de Vallée Poussin kernel for the case of $f(\cdot+iy)\in L^p(\mathbb{R})$, $p\in[1, \infty]$ to solve this problem respectively. Some error estimates were obtained in [9]. However, there were no numerical results. In recent published paper [7,8], a Fourier method and a modified Tikhonov method were proposed, respectively, in which the *a priori* parameter choice rules and the corresponding error estimates were all given. Some numerical examples were implemented to verify the effectiveness of these methods.

In the present paper, we will propose another different regularization method—a mollification method to solve this problem. It is well known that mollification methods are well studied and being widely used as regularization methods in many ill-posed problems [10,11,18]. The ill-posedness of many ill-posed problems is caused by the high-frequency disturbance of the noise data. The basic idea of mollification methods is to use a convolution of the noise data and a smooth function with a parameter to filter the high-frequency components of the noise data, such that the problem becomes well-posed. Furthermore, by proper choice of the parameter, the solution of this well-posed problem can approximate the solution of the original one. Meanwhile, there is a very close connection between mollification methods and approximate inverse method which is also well studied and being used in computerized tomograph and a lot of applications [16,17,20,21]. For simplicity, we assume $A: L^2(\mathbb{R})\to L^2(\mathbb{R})$ be a linear compact operator and its range non-closed. Then the problem of solving $Af=g$ is ill-posed [4,13]. It is well known that for a function $f(x)\in L^2(\mathbb{R})$, there must exist a smooth function $m_\gamma(x)$ (e.g., Gauss or Dirichlet functions, etc.) with parameter γ, such that

$$f_\gamma(x) := \langle f(x'), m_\gamma(x,x')\rangle$$

can approximate *f*(*x*), where $m_\gamma(x,x')=m_\gamma(x-x')$ and $\langle\cdot,\cdot\rangle$ denotes the inner in $L^2(\mathbb{R})$. In the special case that $m_\gamma(x)$ is in the range of A^*, we have, with ψ_γ being the solution of the equation $A^*\psi_\gamma=m_\gamma$ and can be given analytically, the relation

$$f_\gamma(x) := \langle f(\cdot), m_\gamma(x-\cdot)\rangle = \langle f(\cdot), (A^*\psi_\gamma)(x-\cdot)\rangle = \langle (Af)(\cdot), \psi_\gamma(x-\cdot)\rangle = \langle g(\cdot), \psi_\gamma(x-\cdot)\rangle.$$

For the noise data g_δ, it is easy to know that $f_\gamma^\delta := \langle g_\delta(\cdot), \psi_\gamma(x-\cdot)\rangle$ with the reconstruction kernel $\psi_\gamma(x)$ associated with $m_\gamma(x,x')$ is an approximation of the exact solution of the original equation $Af=g$ [13]. This is just the so called approximate inverse method proposed by A.K. Louis [16,21]. Meanwhile, the moment $\langle g_\delta(\cdot), \psi_\gamma(x-\cdot)\rangle$ also implies smoothing the data g_δ, so it is also a mollification method.

In this paper, we only consider a mollification method with the Gauss kernel [18,21] and one of the highlights is that a new *a posteriori* parameter choice rule is given. The comparison of numerical effect between *a priori* and *a posteriori* methods is also provided.

Let $\hat{g}$ denote the Fourier transform of function $g(x)$ defined by

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} g(x)\,\mathrm{d}x, \tag{1.2}$$

and $\|\cdot\|$ denote the norm in $L^2(\mathbb{R})$ defined by

$$\|g\| = \left(\int_{\mathbb{R}} |g(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{1.3}$$

Moreover, suppose $f(\cdot+iy)\in L^2(\mathbb{R})$ for $|y|<y_0$. It is obvious that

$$f(z) = f(x+iy) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(x+iy)\xi}\hat{f}(\xi)\,\mathrm{d}x,$$

i.e.,

$$f(z) = f(x+iy) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi}e^{-y\xi}\hat{f}(\xi)\,\mathrm{d}\xi, \quad |y|<y_0. \tag{1.4}$$