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Quadratic spline quasi-interpolants and collocation methods

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Abstract

Univariate and multivariate quadratic spline quasi-interpolants provide interesting approximation formulas for derivatives of approximated functions that can be very accurate at some points thanks to the superconvergence properties of these operators. Moreover, they also give rise to good global approximations of derivatives on the whole domain of definition. From these results, some collocation methods are deduced for the solution of ordinary or partial differential equations with boundary conditions. Their convergence properties are illustrated and compared with finite difference methods on some numerical examples of elliptic boundary value problems.

MSC: 65D07; 65D25; 65N35

Keywords: Spline approximants; Numerical differentiation; Spline collocation methods

1. Introduction

Univariate and multivariate spline quasi-interpolants (abbr. QIs) have been studied for several decades in approximation theory (see e.g. [3a,c,1,8,9a,b,c,]). However, few studies have been devoted to their potential applications in numerical analysis. Such a program has already been initiated in [7a,b] where some applications to the approximation of derivatives have been developed. In this paper, we focus on approximations of first and second derivatives by those of quadratic spline quasi-interpolants and their applications to collocation methods. Though it is possible to use spline QIs of higher degrees and smoothness, we want to show that already simple C^1 quadratic splines provide rather good numerical methods.

Let $I = [a, b]$ with the uniform partition $\mathcal{X}_n = \{x_i = a + ih,\ 0 \le i \le n\}$ where $h = (b - a)/n$, and $x_{-2} = a - 2h$, $x_{-1} = a - h$, $x_{n+1} = b + h$, $x_{n+2} = b + 2h$. For $1 \le i \le n$, let $t_i = (1/2)(x_{i-1} + x_i)$, $u_i = t_i - hr/6$, $v_i = t_i + hr/6$ (where $r{:=}\sqrt{3}$), and let $t_0 = a$, $t_{n+1} = b$, $v_0 = a$, $u_{n+1} = b$. Let $J{:=}\{0, 1, \ldots, n + 1\}$. On $\mathcal{T}_n{:=}\{t_i,\ i \in J\}$ we define $f_i = f(t_i)$, $0 \le i \le n + 1$, and on the set of *Gauss abscissas* $\mathcal{G}_n{:=}\{v_0; u_i, v_i, 1 \le i \le n; u_{n+1}\}$, we define $\tilde{f}_i = f(u_i)$, $1 \le i \le n + 1$ and $\hat{f}_i = f(v_i)$, $0 \le i \le n$. Quadratic B-splines $\{B_i, i \in J\}$, with supports $[x_{i-2}, x_{i+1}]$ form a basis of the $n + 2$-dimensional space $\mathcal{S}_2(I, \mathcal{X}_n)$ of C^1 quadratic splines (see e.g. [3b,2,10]).

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In Section 2, we study the *uniform quadratic spline quasi-interpolant* (abbr. uniform QI) $\bar{Q}$, and the *Gauss quadratic spline quasi-interpolant* (abbr. Gauss QI) $\tilde{Q}$, respectively defined by

$$\bar{Q}f = \sum_{i \in J} \lambda_i(f) B_i \quad \text{and} \quad \tilde{Q}f = \sum_{i \in J} \mu_i(f) B_i,$$

where the coefficient functionals are given, for $2 \leq i \leq n-1$, by

$$\lambda_i(f) = \frac{1}{8}(-f_{i-1} + 10 f_i - f_{i+1}) \quad \text{and} \quad \mu_i(f) = \alpha(\tilde{f}_i + \hat{f}_i) + \beta(\hat{f}_{i-1} + \tilde{f}_{i+1}),$$

with $\alpha = (9+r)/12$ and $\beta = -(3+r)/12$. For extreme indices, the functionals have specific forms (see Section 2 below). The uniform QI is based on the set $\mathcal{X}_n$ and the Gauss QI is based on the set $\mathcal{G}_n$. The choice of coefficients implies that the two QIs are exact on $\mathbb{P}_2$, i.e. $\bar{Q}m_s = \tilde{Q}m_s = m_s$ for $s = 0, 1, 2$, with the notation $m_s(x) := x^s$. This can be verified by using the B-spline representation of monomials. It implies in particular ([5], chapter 5), that the *global approximation error* $Qf - f$ on a smooth function f is $O(h^3)$.

However, we notice that $Qf(a) = f(a)$, $Qf(b) = f(b)$, $Qf(x_i) - f(x_i) = O(h^4)$, $1 \leq i \leq n-1$, and $Qf(t_i) - f(t_i) = O(h^4)$, $1 \leq i \leq n$, for both quasi-interpolants $Q = \bar{Q}$ or $\tilde{Q}$. Therefore there is a *superconvergence phenomenon* of these operators on the sets of points $\mathcal{X}_n$ and $\mathcal{T}_n$. This phenomenon *does not hold* on $\mathcal{G}_n$. On the other hand, in Section 3, we show that *another superconvergence phenomenon* takes place at Gaussian points for *f* irst derivatives, which leads to an improvement of global approximation properties of these derivatives. Section 4 describes some *derivation matrices* which are used in Section 5 in collocation methods for the solution of some univariate Dirichlet problems (see e.g. [6a,b]). The numerical results show an improvement with respect to classical finite difference methods. Finally, Sections 6 and 7 give some bivariate extensions of previous methods and an application to the Dirichlet problem for the Laplace equation. Once again, the new method is better than finite difference methods for the Laplacian.

2. Univariate quadratic spline quasi-interpolants

2.1. Properties of the uniform quasi-interpolant

The specific coefficient functionals for extreme indices of the *u* niform quadratic spline quasi-interpolant $\bar{Q}f = \sum_{i \in J} \lambda_i(f) B_i$ are defined as follows:

$$\lambda_0(f) := \frac{12}{5} f_0 - \frac{13}{8} f_1 + \frac{1}{4} f_2 - \frac{1}{40} f_3, \quad \lambda_{n+1}(f) := \frac{12}{5} f_{n+1} - \frac{13}{8} f_n + \frac{1}{4} f_{n-1} - \frac{1}{40} f_{n-2},$$
$$\lambda_1(f) := -\frac{2}{5} f_0 + \frac{13}{8} f_1 - \frac{1}{4} f_2 + \frac{1}{40} f_3, \quad \lambda_n(f) := -\frac{2}{5} f_{n+1} + \frac{13}{8} f_n - \frac{1}{4} f_{n-1} + \frac{1}{40} f_{n-2}.$$

We notice that $\bar{Q}f(a) = (\lambda_0 + \lambda_1)/2 = f_0 = f(a)$ and $\bar{Q}f(b) = (\lambda_n + \lambda_{n+1})/2 = f_{n+1} = f(b)$, therefore $\bar{Q}f$ *interpolates* f at the extreme points of I. This QI can also be written in the *quasi-Lagrange form*

$$\bar{Q}f = \sum_{j \in J} f_j \bar{B}_j, \quad \text{with} \quad \bar{B}_j := \frac{1}{8}(-B_{j-1} + 10 B_j - B_{j+1}), \quad 4 \leq j \leq n-3.$$

The coefficients in the basis $\{B_i, i \in J\}$ of the first and last four *quasi-Lagrange functions* are given in the following table (with upper (resp. lower) indices for left (resp. right) functions).

i	0	1	2	3	4	
$\bar{B}_0$	12/5	−2/5	0	0	0	$\bar{B}_{n+1}$
$\bar{B}_1$	−13/8	13/8	−1/8	0	0	$\bar{B}_n$
$\bar{B}_2$	1/4	−1/4	5/4	−1/8	0	$\bar{B}_{n-1}$
$\bar{B}_3$	−1/40	1/40	−1/8	5/4	−1/8	$\bar{B}_{n-2}$
	$n+1$	n	$n-1$	$n-2$	$n-3$	i

Proposition 1. *The quasi-interpolant $\bar{Q}$ is exact on the space $\mathbb{P}_2$ of quadratic polynomials, i.e. $\bar{Q}m_r = m_r$ for $r = 0, 1, 2$. Moreover $\|\bar{Q}\|_\infty = 3/2$.*