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A two-asset stochastic model for long-term portfolio selection

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Abstract

In mean–variance (M–V) analysis, an investor with a holding period $[0,T]$ operates in a two-dimensional space—one is the mean and the other is the variance. At time 0, he/she evaluates alternative portfolios based on their means and variances, and holds a combination of the market portfolio (e.g., an index fund) and the risk-free asset to maximize his/her expected utility at time T. In our continuous-time model, we operate in a three-dimensional space—the first is the spot rate, the second is the expected return on the risky asset (e.g., an index fund), and the third is time. At various times over $[0,T]$, we determine, for each combination of the spot rate and expected return, the optimum fractions invested in the risky and risk-free assets to maximize our expected utility at time T. Hence, unlike those static *M–V* models, our dynamic model allows investors to trade at any time in response to changes in the market conditions and the length of their holding period. Our results show that (1) the optimum fraction $y^*(t)$ in the risky asset increases as the expected return increases but decreases as the spot rate increases; (2) $y^*(t)$ decreases as the holding period shortens; and (3) $y^*(t)$ decreases as the risk aversion parameter-γ is larger.

Keywords: Static mean–variance analysis; Continuous-time formulation; Ornstein–Uhlenbeck process; Holding period

1. Introduction

The portfolio selection problem of an investor is to choose at the beginning of the holding period an optimum portfolio with the objective of maximizing his/her expected utility of wealth at the end of the holding period $[0,T]$, where 0 and T denote the beginning and end of the holding period. The earliest research on solving such portfolio problem, pioneered by Markowitz [21], is the static mean–variance (M–V) analysis in which Markowitz proposes using the mean (also called the expected return) and variance of asset returns as the criteria for portfolio selection. That is, the mean is used as a measure of central tendency and the variance is used as a measure of risk or variation. By focusing only on mean and variance, M–V analysis [11,14,26] assumes that no other statistics are necessary to describe the probability distribution of asset returns.

According to M–V analysis, an investor will choose at time 0 his/her optimum portfolio from a set of portfolios that (i) has the maximum expected return for different levels of variance risk and (ii) has minimum variance risk for different levels of expected return. The set of portfolios satisfying conditions (i) and (ii) is known as the efficient frontier. Under the M–V assumptions, we obtain the two-fund separation principle[1] [9] which says that “an investor will hold at time 0 a utility-maximizing portfolio consisting of a combination of the risk-free asset and a portfolio (often referred to as

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[1] Tobin [28] is credited for developing the two-fund separation principle which extends M–V analysis to include risk-free lending and borrowing.

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the tangency portfolio) of risky assets that is obtained by the straight line emanating from the constant risk-free rate of return and tangent to the efficient frontier." In the limiting case where all risky assets in the market are included in M–V analysis, the tangency portfolio will correspond to what is known as the market portfolio [9,14,26], which, by definition, is a portfolio consisting of all risky assets in the market. In the market portfolio, the proportion invested in each risky asset equals the percentage of the total market capitalization represented by the asset. Now put differently, the two-fund separation principle states that, to maximize his/her expected utility at time T, each investor, based on his/her risk averseness, will decide at time 0 (and only at time 0) to hold a certain combination of two assets: the risk-free asset and the market portfolio.[2]

However, a close examination of M–V models for portfolio selection highlights two deficiencies. The first is the static nature of the models.[3] That is, M–V models assume that the rate of return (or the spot rate) on the risk-free asset and the means and variances of asset returns do not vary over the holding period. It is not too erroneous to assume that they remain unchanged for short holding period (e.g., 3 months or 6 months), but it is quite inaccurate to assume that they remain unchanged for long holding period (e.g., 5 years or longer). For long holding period, an M–V model under such assumption can hardly provide an adequate description of the actual securities market. If there are dramatic changes in asset prices before time T, M–V models are not equipped with the proper mechanism for timely portfolio adjustments by the investor. The second is the assumption of constant rate of return for the risk-free asset. As always, interest rates change constantly and unpredictably. Hence, this assumption is inconsistent with reality. In fact, numerous empirical studies have attested the dynamic and volatile nature of asset prices [8,10,12,19] and interest rates [20,24,30]. Hence, portfolio selection based on M–V analysis is bound to err for long holding period.

This study proposes an approach to deal with the two deficiencies. First, we use continuous-time mathematics to handle dynamic portfolio selection problems. In our continuous-time framework, the securities market is modeled with many trading dates to accommodate trading activities at different times over the holding period. In particular, we use stochastic differential equations with drift and volatility to depict the dynamic and volatile nature of the prices of the two assets: the risk-free asset[4] and the market portfolio (or simply the risky asset). Second, to characterize the stochastic nature of interest rates, we assume that the spot rate follows an Ornstein–Uhlenbeck (O–U) process. The reason for using this process to characterize the dynamics of the spot rate is its mean-reverting property. The mean reversion[5] in the O–U process is such that the spot rate will tend to be pulled back to some long-run average level over time when it is either too high or too low.

Hence, in our continuous-time framework, each investor, based on his/her degree of risk averseness, determines at various times over the holding period the optimum allocation between the risky[6] and risk-free assets to maximize his/her expected utility at time T. In this sense, our model is superior to those M–V models in that ours are dynamic but theirs are static. To sum up, in the static M–V analysis, an investor operates in a two-dimensional space—one is the mean and the other is the variance. At time 0 (and only at time 0), he/she evaluates alternative portfolios, based on their means and variances, and holds a combination of the risky and risk-free assets to maximize his/her expected utility at time T. In sharp contrast, in our continuous-time formulation, an investor operates in a three-dimensional space—the first is the spot rate, the second is the expected return on the risky asset, and the third is time. At different times over the holding period, he/she evaluates alternative portfolios, based on each combination of the spot rate and expected return, and makes the optimum allocation of funds between the risky and risk-free assets so as to maximize his/her expected utility at terminal time T.

The rest of the paper proceeds as follows. In Section 2, we use continuous-time mathematics to set up a two-asset stochastic model and derive an explicit formula for the optimum fraction of wealth invested in the risky and risk-free assets at each time t. In Section 3, we use maximum likelihood method to estimate the relevant parameters of our model and implement it using backward recursion. In Section 4, we report the optimum fractions invested in the risky asset

[2] In practice, instead of holding a combination of the risk-free asset and market portfolio, an investor will decide at time 0 to hold a certain combination of the risk-free asset (such as U.S. T-bill) and a broad, well-diversified portfolio (such as an index fund).

[3] Many studies [7,18,23,25,27] propose other approaches for portfolio selection. However, these other approaches are all static in nature.

[4] In the context of M–V analysis, a default-free security with a maturity that matches the length of the holding period is considered a risk-free asset. For example, a 3-month U.S. T-bill is a risk-free asset for a 3-month holding period, but not for a 1-year holding period.

[5] There are good economic justifications in favor of mean reversion. When interest rates are high, the demand for funds will decrease and, as a result, they will decline. When interest rates are low, the demand for funds will increase and, as a result, they will rise.

[6] Note that the risky asset in our model represents the market portfolio or a broad, well-diversified market index (e.g., the S&P 500).