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Errors of linear multistep methods for singularly perturbed Volterra delay-integro-differential equations ☆

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Abstract

This paper is concerned with the error behaviour of linear multistep methods applied to singularly perturbed Volterra delay-integro-differential equations. We derive global error estimates of $A(\alpha)$-stable linear multistep methods with convergent quadrature rule. Numerical experiments confirm our theoretical analysis.

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Keywords: Singularly perturbed systems; Volterra delay-integro-differential equations; Linear multistep methods; Error analysis; Quadrature rule

1. Introduction

Singularly perturbed problems (SPPs) arise in many physical and biological problems (cf. [1–3,6,9–11,13]). It is well known that the convergence of numerical methods for SPPs cannot be satisfactorily covered by B-theory because of their very special structures. Some authors (cf. [12,18–21]) have presented some numerical convergence results of SPPs. A special class of singularly perturbed integro-differential systems has been solved by Kauthen [14,15] by implicit Runge–Kutta. In Gan et al. [8], convergence of linear multistep methods and Runge–Kutta methods applied to the singular perturbation problems with delays has been analyzed. Hristova and Bainov [9–11] investigated some character of singularly perturbed Volterra delay-integro-differential equations (SPVDIDEs).

As far as we know, no results on the convergence for the systems of SPVDIDEs have been presented in the literature. In this paper, we analyze the following systems of SPVDIDEs

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$$\begin{cases} x'(t) = f\left(t, x(t), y(t), \int_{t-\tau}^{t} p(t, s, x(s), y(s))\, ds\right), & t \in [0, T], \\ \epsilon y'(t) = g\left(t, x(t), y(t), \int_{t-\tau}^{t} q(t, s, x(s), y(s))\, ds\right), & 0 < \epsilon \ll 1, t \in [0, T], \\ x(t) = \varphi(t), \quad y(t) = \psi(t), & t \leq 0, \end{cases} \tag{1.1}$$

where τ and ϵ are constants, and $\tau > 0$. φ and ψ are given continuous functions. $p : R \times R \times R^{D_1} \times R^{D_2} \to R^{D_3}$, $f : R \times R^{D_1} \times R^{D_2} \times R^{D_3} \to R^{D_1}$, $q : R \times R \times R^{D_1} \times R^{D_2} \to R^{D_4}$ and $g : R \times R^{D_1} \times R^{D_2} \times R^{D_4} \to R^{D_2}$ are given, sufficiently smooth functions (let us say of class C^m with sufficiently large m). $x : R \to R^{D_1}$ and $y : R \to R^{D_2}$ are solutions of the system (1.1). In order to make the error analysis feasible, we always assume that system (1.1) has a unique solution $(x(t), y(t))$ which is sufficiently differentiable and satisfies

$$\left|\frac{d^i x(t)}{dt^i}\right| \leq M_i, \left|\frac{d^i y(t)}{dt^i}\right| \leq N_i,$$

where M_i and N_i are constants which are independent of the stiffness of the problem. And the system (1.1) satisfies the conditions

$$\|f(t, x_1, y_1, u_1) - f(t, x_2, y_2, u_2)\| \leq \beta\|x_1 - x_2\| + \sigma_1\|y_1 - y_2\| + \sigma_2\|u_1 - u_2\|, \quad t \geq 0 \tag{1.2a}$$

$$\|p(t, s, x_1, y_1) - p(t, s, x_2, y_2)\| \leq \gamma_1\|x_1 - x_2\| + \gamma_2\|y_1 - y_2\|, \quad t \geq 0 \tag{1.2b}$$

$$\|g(t, x_1, y, v_1) - g(t, x_2, y, v_2)\| \leq \sigma_3\|x_1 - x_2\| + \sigma_4\|v_1 - v_2\|, \quad t \geq 0 \tag{1.2c}$$

$$\|q(t, s, x_1, y_1) - q(t, s, x_2, y_2)\| \leq \gamma_3\|x_1 - x_2\| + \gamma_4\|y_1 - y_2\|, \quad t \geq 0 \tag{1.2d}$$

where $\langle, \rangle$ is the standard inner product on R^N and $\|\cdot\|$ the corresponding norm. For reasons of stability we assume that

$$\text{the eigenvalues} \lambda \text{of} g_y(t, x, y, v) \text{lie in} |arg\lambda - \pi| < \alpha, \tag{1.2e}$$

for (t, x, y, v) in a neighbourhood of the considered solution, except for special instructions, where α is the angle of $A(\alpha)$-stability of the following linear multistep methods, $arg\lambda$ is the argument of the eigenvalue λ.

This paper is concerned with the error analysis of linear multistep methods applied to this classes of SPVDIDEs which satisfy (1.2). This paper is organized as follows. In Section 2, we drive the linear multistep methods with compound quadrature formulae. In Section 3, the global error estimate of $A(\alpha)$-stable and strictly stable multistep method is investigated. In Section 4, we illustrate our main results by numerical experiments.

2. Linear multistep methods for SPVDIDEs

Baker and Ford[4,5] derived a class of numerical methods for Volterra delay-integro-differential equations (VDIDEs) with discrete delay arguments. Their methods are based on strongly stable underlying linear multistep methods combined with convergent quadrature rules. Here, we introduce an adaptation of those methods to (1.1):

$$\rho(E)x_n = h\sigma(E)f(t_n, x_n, y_n, u_n), \tag{2.1a}$$

$$\rho(E)y_n = \frac{h}{\epsilon}\sigma(E)g(t_n, x_n, y_n, v_n), \tag{2.1b}$$

where $\rho(\xi) = \sum_{i=0}^{k}\alpha_i\xi^i$, $\sigma(\xi) = \sum_{i=0}^{k}\beta_i\xi^i$, $(\alpha_i, \beta_i \in R)$ are polynomials, which are subject to consistency conditions $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$, E denotes the shift operator: $Ey_n = y_{n+1}$, $h > 0$ is the stepsize, $t_n = nh, n = 0, 1, \ldots, I, (I + k)h \leq T, x_n$ and y_n are approximations to the exact solution $x(t_n)$ and $y(t_n)$, respectively. The arguments u_n and v_n are approximations to integrals $\int_{t_n-\tau}^{t_n} p(t_n, s, x(s), y(s))\, ds$ and $\int_{t_n-\tau}^{t_n} q(t_n, s, x(s), y(s))\, ds$, respectively.

Process (2.1) is defined completely by the linear multistep method and the quadrature rules for u_n and v_n.