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Incremental unknowns method based on the θ-scheme for time-dependent convection–diffusion equations

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Abstract

A θ-scheme using two-level incremental unknowns is presented for solving time-dependent convection–diffusion equations in two-dimensional case. The IMG algorithm (Inertial Manifold–Multigrid algorithm) including the second-order incremental unknowns is convergent. The incremental unknowns method based on the θ-scheme needs a stability condition as $0 \le \theta < 1/2$ and is unconditionally stable as $1/2 \le \theta \le 1$. By the GMRES method in the iteration at each time step, numerical results of the convection–diffusion equations are investigated and confirm that oscillations can be controlled by the incremental unknowns method.

AMS Subject Classification: 65N05

Keywords: Incremental unknowns; θ-Scheme; Convection–diffusion equation

1. Introduction

As we know, for convection dominated diffusion equations, since the diffusion coefficients are often much smaller than transport velocity, it is very difficult to simulate the convection–diffusion problems numerically. The classical finite difference and Galerkin methods introduce nonphysical oscillations into the numerical solutions [3,19,25]. Many different schemes to overcome the deficiencies of the Galerkin approach in highly convective situations have been developed: the wavelet and multigrid method [15], the hierarchical basis method [1], the multiscale method [18], the adaptive finite element method [10,17], the θ-scheme [21] and some compact schemes [20,22]. The upwind scheme avoids the numerical oscillation, but it only has the first-order accuracy [12,16]. Some modified upwind schemes were proposed by introducing the modified second-order differential terms to obtain the second-order accuracy, but some of them did not preserve numerical stability for a long time integration [23]. Thus, there is considerable interest in forming good schemes that overcome numerical oscillations and studying the long-term dynamic behavior of convection–diffusion equations.

Some efficient preconditioners have been presented in [6,11,15] for solving convection–diffusion equations. The preconditioner based on the hierarchical basis on nested grids is known as incremental unknowns (IU) in [5,7,28] when several levels of finite difference discretization are considered. The incremental unknowns method has been

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successfully used in fluid mechanics problem, such as the application in Burgers equations [4,8]. For computational fluid problems containing shocks and the corresponding conservative schemes, we refer the reader to [9,27] and the references therein.

In finite differences case, the IMG algorithm (Inertial Manifold–Multigrid algorithm) including the first-order and second-order incremental unknowns was introduced in [5]. A partial justification of the IMG algorithm [13,29] lays in the fact that some quantities in the usual finite difference methods are small and could be neglected. Inertial and approximate inertial manifolds correspond to an exact (or approximate) iteration law between small and large wavelengths. It is natural to decompose the unknown function into its long wavelength and its short wavelength components, which have been realized by Temam using the first-order and second-order incremental unknowns in [5,29]. Moreover, they are particularly adapted to the integration of evolutionary equations on large intervals of time [30].

The incremental unknowns method is a IMG algorithm based on finite difference discretization. The order of the incremental unknowns does not influence significantly the convergence behavior of the iterative process that is fast and smooth [14]. Hereafter, we will investigate convergence of the IMG algorithm and stability of the two-level incremental unknowns method in a fully discrete θ-scheme for solving time-dependent convection–diffusion equations. Importantly, the incremental unknowns method can be used to control numerical oscillations, however, the classical θ-scheme [2] cannot attain the point.

This paper is organized as follows. In Section 2, the θ-scheme with the use of IU is given for the convection–diffusion problem. In Section 3, the aim is to construct a convergent IMG algorithm. In Section 4, a priori estimates of the incremental unknowns are presented. Stability of the incremental unknowns method based on the θ-scheme are discussed in Section 5. Finally, some numerical results confirm efficiency of the incremental unknowns method.

2. Incremental unknowns and corresponding numerical schemes

In this section we will introduce second-order incremental unknowns and present the corresponding numerical schemes for solving the two-dimensional time-dependent convection–diffusion equation with the Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + a_1\dfrac{\partial u}{\partial x} + a_2\dfrac{\partial u}{\partial y} = f, & \text{in } \Omega \\ u(x, y, t) = 0, & \text{on } \partial\Omega \times [0, T] \\ u(x, y, 0) = u_0(x, y), & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here a_1, a_2 are given constants, $\nu > 0$, and Ω is a square $(a, b)^2$ in $\mathbb{R}^2$. We are mainly interested in the case $|a_1|, |a_2| \gg \nu$.

Let $N \in \mathbb{N}$, we introduce the fine grid corresponding to the mesh size $h = (b - a)/2N$ and the coarse grid corresponding to the mesh size $2h = (b - a)/N$ in both directions. For $i, j = 0, \ldots, 2N$, we write $f_{ij} = f(x_i, x_j)$ and $u_{ij} \simeq u(x_i, x_j)$ that is the approximate value of $u(x_i, x_j)$, with taking $x_i = a + ih$, $y_j = a + jh$. The nodes of the coarse grid are the points (x_{2i}, x_{2j}), $i = 1, \ldots, N - 1$, $j = 1, \ldots, N - 1$, and the nodes of the fine grid are those points (x_i, x_j), $i = 1, \ldots, 2N - 1$, $j = 1, \ldots, 2N - 1$. We reorder the nodal unknowns U in the hierarchical way that the nodal unknowns U_c on the coarse grid first are numbered from left to right and from top to bottom, and then the nodal unknowns U_f on the fine grid that do not belong to the coarse grid are numbered similarly.

Let A, B_1, B_2 be the underlying matrices corresponding to the central finite difference discretization of differential operators $-\Delta$, $(\partial/\partial x)$, $(\partial/\partial y)$, respectively,

$$-\Delta u_{i,j} = \frac{1}{h^2}(4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}), \tag{2.2}$$

$$\frac{\partial}{\partial x}u_{i,j} = \frac{1}{2h}(u_{i+1,j} - u_{i-1,j}), \tag{2.3}$$

$$\frac{\partial}{\partial y}u_{i,j} = \frac{1}{2h}(u_{i,j+1} - u_{i,j-1}), \tag{2.4}$$