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Initial-boundary value problems of warped MPDAEs including minimisation criteria

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Abstract

Electric circuits, which produce oscillations at widely separated time scales, cause a huge computational effort in a numerical simulation of the mathematical model based on differential-algebraic equations (DAEs). Alternatively, a multidimensional signal model yields a description via multirate partial differential-algebraic equations (MPDAEs). Initial-boundary value problems of the MPDAE system reproduce solutions of the underlying DAE system. In case of frequency modulation, an additional function occurs in the MPDAE model, which represents a degree of freedom in the multivariate description of the signals. We present two minimisation strategies, which are able to identify the additional parameters such that the resulting solutions exhibit an elementary structure. Thus numerical schemes can apply relatively coarse grids and an efficient simulation is achieved.

Keywords: Multirate partial differential-algebraic equation; Initial-boundary value problem; Variational calculus; Method of lines; Frequency modulation

1. Introduction

Mathematical modelling of electric circuits applies a network approach, which generates systems of differential-algebraic equations (DAEs), see [3]. In many applications, signals act on widely separated time scales. For example, fast oscillations may exhibit amplitude as well as frequency modulation, which are caused by slowly varying parts. Thus a numerical integration of the circuit's DAEs becomes inefficient, since the high-frequency oscillations limit the size of time steps, whereas the slow time scales imply a relatively long time interval of the simulation.

Alternatively, a multivariate signal model is able to decouple the time scales. Based on multivariate functions, Brachtendorf et al. [1] transformed the DAE system into multirate partial differential-algebraic equations (MPDAEs). Accordingly, a solution of the MPDAE system reproduces a solution of the underlying DAE system. Initial-boundary value problems or multiperiodic boundary value problems of the MPDAEs are considered to obtain corresponding solutions. The multidimensional approach has been successfully used for simulating circuits with amplitude modulated signals, see [11].

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In case of frequency modulated signals, the model has to incorporate a local frequency function to achieve an efficient representation. Narayan and Roychowdhury [7] introduced a corresponding system of warped MPDAEs, which includes these parameters. Thereby, the local frequency function represents a degree of freedom in the modelling. Inappropriate choices cause many oscillations in the multivariate model and thus the strategy becomes inefficient. Hence additional conditions are necessary, which identify adequate local frequency functions.

An elementary approach consists in demanding continuous phase conditions, see [7]. Such additional constraints have been used successfully in time domain, see [8], as well as in frequency domain, see [12]. However, the phase conditions operate just on one component of the solution. Thus the idea is to impose minimisation demands, which guarantee an elementary structure in each component of the multivariate functions. Houben [6] constructed a strategy for solving initial-boundary value problems, where the amount of certain partial derivatives is minimised locally. On the other hand, a global minimisation criterion for determining multiperiodic solutions is introduced in [9].

In this paper, we present two strategies based on minimisation for solving initial-boundary value problems of the MPDAE system. The first approach is a direct generalisation of the technique given in [6], where a weighted norm is applied now. This strategy minimises the amount of certain partial derivatives corresponding to a charge term. Alternatively, the second technique demands a minimisation criterion for according derivatives of the solution itself. A necessary condition for an optimal solution results from a variational calculus with respect to transformation properties of solutions. We discuss advantages and disadvantages of both approaches. If the circuit's equations represent a system of ordinary differential equations (ODEs), then the two techniques are equivalent.

The paper is organised as follows. In Section 2, we sketch the multivariate signal model and the resulting MPDAE model. Thereby, required transformation properties of solutions are discussed. We derive the two minimisation criteria for identifying adequate solutions in Section 3. A numerical scheme based on a method of lines, which allows for including the conditions achieved from the minimisation criteria, is constructed in Section 4. Finally, we present according numerical results using two test examples.

2. MPDAE model

A multidimensional signal model decouples widely separated time scales in radio frequency signals. Consequently, we obtain a mathematical model of time-dependent systems, where several variables represent the different rates.

2.1. Multivariate signal model

We start with an outline of the multidimensional signal model. Firstly, we consider amplitude modulated signals. For example, the time-dependent function

$$y(t) := \left[1 + \alpha \sin\left(\frac{2\pi}{T_1} t\right)\right] \sin\left(\frac{2\pi}{T_2} t\right) \tag{1}$$

with $T_1 \gg T_2$ represents a high-frequency oscillation, where a fixed parameter $\alpha \in (0, 1)$ introduces a modulation by a low-frequency oscillation, see Fig. 1 (left). Therefore we require many time steps to resolve all oscillations in case of widely separated rates. Nevertheless, we can introduce an own variable for each time scale, which yields the formulation

$$\hat{y}(t_1, t_2) := \left[1 + \alpha \sin\left(\frac{2\pi}{T_1} t_1\right)\right] \sin\left(\frac{2\pi}{T_2} t_2\right). \tag{2}$$

This representation is called the *multivariate function* (MVF) of the signal (1). The MVF (2) is biperiodic and thus just determined by its values in the rectangle $[0, T_1] \times [0, T_2]$. Fig. 1 (right) illustrates that the MVF exhibits a simple structure. Hence a coarse grid in time domain is sufficient for resolving this representation. We can completely reconstruct the original signal (1) via $y(t) = \hat{y}(t, t)$. This approach achieves an efficient multidimensional model of amplitude modulated signals.

Secondly, we examine signals, which include amplitude modulation as well as frequency modulation. In the signal

$$x(t) := \left[1 + \alpha \sin\left(\frac{2\pi}{T_1} t\right)\right] \sin\left(\frac{2\pi}{T_2} t + \beta \cos\left(\frac{2\pi}{T_1} t\right)\right), \tag{3}$$