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Lifted, projected and subgraph-induced inequalities for the representatives k-fold coloring polytope

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A B S T R A C T

A k-fold x-coloring of a graph G is an assignment of (at least) k distinct colors from the set $\{1, 2, \ldots, x\}$ to each vertex such that any two adjacent vertices are assigned disjoint sets of colors. The kth chromatic number of G, denoted by $\chi_k(G)$, is the smallest x such that G admits a k-fold x-coloring. We present an integer linear programming formulation (ILP) to determine $\chi_k(G)$ and study the facial structure of the corresponding polytope $\mathcal{P}_k(G)$. We show facets that $\mathcal{P}_{k+1}(G)$ inherits from $\mathcal{P}_k(G)$ and show how to lift facets from $\mathcal{P}_k(G)$ to $\mathcal{P}_{k+\ell}(G)$. We project facets of $\mathcal{P}_1(G \circ K_k)$ into facets of $\mathcal{P}_k(G)$, where $G \circ K_k$ is the lexicographic product of G by a clique with k vertices. In both cases, we can obtain facet-defining inequalities from many of those known for the 1-fold coloring polytope. We also derive facets of $\mathcal{P}_k(G)$ from facets of stable set polytopes of subgraphs of G. In addition, we present classes of facet-defining inequalities based on strongly χ_k-critical webs and antiwebs, which extend and generalize known results for 1-fold coloring. We introduce this criticality concept and characterize the webs and antiwebs having such a property.

1. Introduction

All graphs considered here are finite, undirected and simple. Furthermore, for every graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A *k-fold x-coloring* of a graph G is an assignment of (at least) k distinct colors from the set $[x] := \{1, 2, \ldots, x\}$ to each vertex such that any two adjacent vertices are assigned disjoint sets of colors. The kth chromatic number of G, denoted $\chi_k(G)$, is the smallest value x such that G admits a k-fold x-coloring [1]. Obviously, $\chi_1(G) = \chi(G)$ is the conventional *chromatic number*. Additionally, one may easily verify that $\chi_k(G) \leq k \cdot \chi(G)$ for all $k \in \mathbb{N}$. Observe that $\chi_k(G)$ is equivalently defined if each vertex is restricted to receive *exactly* k colors.

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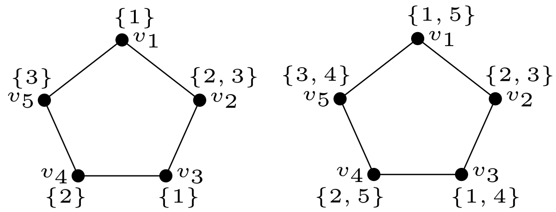

(a) 1-fold 3-coloring. (b) 2-fold 5-coloring.

Fig. 1. Examples of k-fold x-colorings of a cycle C_5.

In Fig. 1, we show k-fold x-colorings of an odd cycle C_5 with 5 vertices. We have a 1-fold 3-coloring (Fig. 1(a)) and a 2-fold 5-coloring (Fig. 1(b)). Actually, they are optimal colorings for $k = 1$ and $k = 2$, respectively, as $\chi_1(C_5) = 3$ and $\chi_2(C_5) = 5$ [2]. In Fig. 1(a), note that we could have removed color 3 from v_2 since it is assigned more than $k = 1$ colors.

The k-fold coloring problem is that of finding a coloring with $\chi_k(G)$ colors. There is a vast literature about the (1-fold) coloring problem, comprising several aspects of the problem such as computational complexity, combinatorial properties, polyhedral studies and solution methods. Recall that determining $\chi(G)$ is an $\mathcal{NP}$-hard problem. In fact, it is $\mathcal{NP}$-hard to approximate the chromatic number within $n^{1-\varepsilon}$ on n-vertex graphs, for every $\varepsilon > 0$ [3]. As a direct consequence, for each fixed $k \in \mathbb{N}$, it is $\mathcal{NP}$-hard to approximate $\chi_k(G)$ within $\frac{n^{1-\varepsilon}}{k}$ on n-vertex graphs G, for every $\varepsilon > 0$.

In comparison with the 1-fold case, there are only a few studies regarding the k-fold coloring. From the theoretical point of view, one of the reasons for this situation could be the fact that a k-fold coloring of a graph G can be given by an 1-fold coloring of the graph $G \circ K_k$, the lexicographic product of G by a clique with k vertices. More precisely, $\chi_k(G) = \chi_1(G \circ K_k)$. Recall that $G \circ K_k$ is obtained by replacing each vertex of G by the clique K_k and making two vertices in $G \circ K_k$ adjacent whenever the corresponding vertices in G are adjacent.

Considering the computational point of view, this approach, however, may suffer from many drawbacks, mainly because $G \circ K_k$ is k times larger than G and the k vertices associated with each vertex of G play essentially the same role in the coloring $G \circ K_k$. In order to make a solution method for finding an 1-fold optimal coloring efficient to the k-fold case, one should have to explore the special structure of $G \circ K_k$ to compensate the increase in the number of vertices. Moreover, in an enumerative procedure, the many symmetric solutions arising from the multiple copies of each vertex of G in $G \circ K_k$ may cause serious disadvantages.

As an alternative, one could try to the tackle the k-fold coloring problem directly (on G). However, it does not seem reasonable to disregard all results about the conventional coloring. This work contributes in this direction and focus on polyhedral studies. For studies on the k-fold coloring problem from the perspective of structural graph theory, the reader is referred to [1,2,4–7].

First, in Section 2, we propose an ILP formulation for the kth fold coloring problem based on class representatives. Class representatives were introduced in [8,9] to model the vertex coloring problem, and since then have been used to model several other problems that aim to cluster elements of a universe according to some property (see e.g. [10–13]). For $k \geq 2$, the formulation we propose for a graph G is more compact than the (1-fold) representatives formulation for the lexicographic product $G \circ K_k$. They coincide when $k = 1$. In the remainder of this paper, we study the polytope $\mathcal{P}_k(G)$ associated with this formulation, which is defined in Section 3.

We profit from the known facial studies for $\mathcal{P}_1(G)$ in two ways. In Section 4, we present a lifting theorem that combines a facet of $\mathcal{P}_k(G)$ and a facet of $\mathcal{P}_\ell(G)$ into a facet of $\mathcal{P}_{k+\ell}(G)$ as well as we show specific liftings from facets of $\mathcal{P}_k(G)$ to facets of $\mathcal{P}_{k+1}(G)$. In Section 5, we show how to project facets from $\mathcal{P}_1(G \circ K_k)$