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The constant objective value property for multidimensional assignment problems

Ante Ćustić*, Bettina Klinz

Institut für Optimierung und Diskrete Mathematik, TU Graz, Steyrergasse 30, A-8010 Graz, Austria

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ABSTRACT

An instance of a combinatorial optimization problem is said to have the constant objective value property (COVP) if every feasible solution has the same objective function value. In this paper our goal is to characterize the set of all instances with the COVP for multidimensional assignment problems.

Our central result deals with planar d-dimensional assignment problems. We show that such constant objective value instances are characterized by so-called sum-decomposable arrays with appropriate parameters. This adds to the known result for the axial d-dimensional case.

1. Introduction

Consider combinatorial optimization problems of the following type. We are given a ground set $E = \{1, \dots, n\}$, a real cost vector $C = (c(1), \dots, c(n))$ and a set of feasible solutions $\mathcal{F} \subseteq 2^{\{1,\dots,n\}}$. The *objective value* of a feasible solution $F \in \mathcal{F}$ is given by the so-called *sum objective function*

$$c(F) := \sum_{i \in F} c(i).$$

The goal is to find a feasible solution F^* such that $c(F^*)$ is minimal. The traveling salesman problem, the linear assignment problem, the shortest path problem, Lawler's quadratic assignment problem and many other well-known combinatorial optimization problems fall into this class of problems.

Definition 1.1. We say that an instance of a combinatorial optimization problem has the *constant objective value property* (COVP) if every feasible solution has the same objective value.

* Corresponding author.
E-mail addresses: custic@opt.math.tugraz.at (A. Ćustić), klinz@opt.math.tugraz.at (B. Klinz).

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In this paper we are interested in characterizing the set of instances with the COVP for the class of multidimensional assignment problems.

Related results. The constant objective value property is closely connected to the notion of *admissible transformations* introduced in 1971 by Vo-Khac [1].

Definition 1.2. A transformation T of the cost vector C to the new cost vector $\tilde{C} = (\tilde{c}(1), \tilde{c}(2), \ldots, \tilde{c}(n))$ is called admissible with index $z(T)$, if

$$c(F) = \tilde{c}(F) + z(T) \quad \text{for all } F \in \mathcal{F}.$$

Note that admissible transformations preserve the relative order of the objective values of all feasible solutions. It is well known that admissible transformations can be used as optimality criterion and to obtain lower bounds which are useful for hard combinatorial optimization problems. Indeed, consider the combinatorial optimization problem $\min_{F \in \mathcal{F}} c(F)$. Let T be an admissible transformation with index $z(T)$ from the original cost vector C to the new cost vector $\tilde{C}$ such that there exists a feasible solution F^* with the following properties:

(i) $\tilde{c}(i) \geq 0$ for all $i \in \{1, \ldots, n\}$,
(ii) $\tilde{c}(F^*) = 0$.

Then F^* is an optimal solution with objective value $z(T)$. If the condition (ii) is not satisfied or we cannot prove that it holds, then $z(T)$ provides a lower bound for the optimal objective value. For the class of combinatorial optimization problems with sum objective function there is a one-to-one correspondence between admissible transformations that transform the cost vector $(c(1), \ldots, c(n))$ to $(\tilde{c}(1), \ldots, \tilde{c}(n))$, and cost vectors $B = (b(1), b(2), \ldots, b(n))$ that fulfill the COVP. The correspondence is obtained by $c(i) = \tilde{c}(i) + b(i)$ for all i. Then the index of the corresponding admissible transformation is $z(T) = \sum_{i \in F} b(i)$ for any $F \in \mathcal{F}$. The correspondence between the COVP and admissible transformations provides a further source of motivation for investigating COVP characterizations.

The COVP characterization for the linear assignment problem can be derived from the work of Berenguer [2], in which he characterized the set of all admissible transformations for the traveling salesman problem (TSP) as well as for the multiple salesmen version. All admissible transformations for the TSP are obtained by adding real values to rows and columns of the distance matrix. In view of the correspondence mentioned above this result can be rephrased as a result on the COVP for the TSP as follows (this has been noted already by Gilmore, Lawler and Shmoys [3]).

An $n \times n$ real matrix $C = (c_{ij})$ is called *sum matrix* if there exist two real n-dimensional vectors $U = (u_i)$ and $V = (v_i)$ such that

$$c_{ij} = u_i + v_j \quad \text{for all } i, j \in \{1, \ldots, n\}. \tag{1}$$

Matrix C is called a *weak sum matrix* if C can be turned into a sum matrix by appropriately changing the entries on its main diagonal. (For the TSP the diagonal entries of C do not play a role and can be ignored.)

Theorem 1.1 (*Berenguer [2], Gabovich [4], Gilmore et al. [3]*). *A TSP instance with the $n \times n$ distance matrix $C = (c_{ij})$ has the COVP if and only C is a weak sum matrix.*

Berenguer's proof works for the linear assignment problem as well, hence, an instance of the linear assignment problem with cost matrix $C = (c_{ij})$ has the COVP if and only if C is a sum matrix. Kaveh [5] generalized this result by proving that instances of the so-called axial d-dimensional assignment problem