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An $\mathcal{O}(n\sqrt{m})$ algorithm for the weighted stable set problem in {claw, net}-free graphs with $\alpha(G) \geq 4$

Paolo Nobili[a], Antonio Sassano[b,*]

[a] *Dipartimento di Ingegneria dell'Innovazione, Università del Salento, Lecce, Italy*

[b] *Dipartimento di Ingegneria Informatica, Automatica e Gestionale "Antonio Ruberti", Università di Roma "Sapienza", Roma, Italy*

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ABSTRACT

In this paper we show that a connected {claw, net}-free graph $G(V, E)$ with $\alpha(G) \geq 4$ is the union of a *strongly bisimplicial* clique Q and at most two *clique-strips*. A clique is strongly bisimplicial if its neighborhood is partitioned into two cliques which are mutually non-adjacent and a clique-strip is a sequence of cliques $\{H_0, \ldots, H_p\}$ with the property that H_i is adjacent only to H_{i-1} and H_{i+1}. By exploiting such a structure we show how to solve the *Maximum Weight Stable Set Problem* in such a graph in time $\mathcal{O}(|V|\sqrt{|E|})$, improving the previous complexity bound of $\mathcal{O}(|V||E|)$.

1. Introduction

The *Maximum Weight Stable Set Problem (MWSSP)* in a graph $G(V, E)$ with node-weight function $w : V \to \Re$ asks for a subset S^* of pairwise non-adjacent *nodes* in V having maximum weight $\sum_{v \in S^*} w(v) = \alpha_w(G)$. For each subset W of V we denote by $\alpha_w(W)$ the maximum weight of a stable set in W. If w is the vector of all 1's we omit the reference to w and write $\alpha(G)$ and $\alpha(W)$.

For each graph $G(V, E)$ we denote by $V(F)$ the set of end-nodes of the edges in $F \subseteq E$, by $E(W)$ the set of edges with end-nodes in $W \subseteq V$ and by $N(W)$ (*neighborhood* of W) the set of nodes in $V \setminus W$ adjacent to some node in W. If $W = \{w\}$ we simply write $N(w)$. We denote by $N[W]$ and $N[w]$ (*closed neighborhood*) the sets $N(W) \cup W$ and $N(w) \cup \{w\}$ and by $\delta(W)$ the set of edges having exactly one end-node in W; if $\delta(W) = \emptyset$ and W is minimal with this property we say that W is (or induces) a connected component of G. We denote by $G - F$ the subgraph of G obtained by removing from G the edges in $F \subseteq E$. A *clique* is a complete subgraph of G induced by some set of nodes $K \subseteq V$. With a little abuse of notation we also regard the set K as a clique and, for any edge $uv \in E$, both uv and $\{u, v\}$ are said to be a clique. A node w such

* Corresponding author.
E-mail address: sassano@dis.uniroma1.it (A. Sassano).

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that $N(w)$ is a clique is said to be *simplicial*. By extension, a clique K such that $N(K)$ is a clique is also said to be *simplicial*. A *claw* is a graph with four nodes w, x, y, z with w adjacent to x, y, z and x, y, z mutually non-adjacent. To highlight its structure, it is denoted as $(w : x, y, z)$. A P_k is a (chordless) path induced by k nodes and will be denoted as $(u_1, \dots, u_k)$. A subset $T \in V$ is *null* (*universal*) to a subset $W \subseteq V \setminus T$ if and only if $N(T) \cap W = \emptyset$ ($N(T) \cap W = W$). Two nodes $u, v \in V$ are said to be *twins* if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. We can always remove a twin from V without affecting the value of the optimal solution of MWSSP. In fact, if $uv \in E$ we can remove the twin with minimum weight, while if $uv \notin E$ we can remove u and replace $w(v)$ by $w(u) + w(v)$. The complexity of finding all the twins is $\mathcal{O}(|V| + |E|)$ [1,2] and hence we assume throughout the paper that our graphs have no twins. A *net* $(x, y, z : x', y', z')$ is a graph induced by a triangle $T = \{x, y, z\}$ and three mutually non-adjacent nodes $\{x', y', z'\}$ with $N(x') \cap T = \{x\}$, $N(y') \cap T = \{y\}$ and $N(z') \cap T = \{z\}$. A *square* is a 4-hole (v_1, v_2, v_3, v_4) with $v_1v_3, v_2v_4 \notin E$ called *diagonals*.

The family of {claw, net}-free graphs has been widely studied in the literature [3–5] since such graphs constitute an important subclass of claw-free graphs. In particular, in [3] Pulleyblank and Shepherd described both a $\mathcal{O}(|V|^4)$ algorithm for the maximum weight stable set problem in distance claw-free graphs (a class containing {claw, net}-free graphs) and the structure of a polyhedron whose projection gives the stable set polyhedron $STAB(G)$. In [6] Faenza, Oriolo and Stauffer reduced the complexity of MWSSP in {claw, net}-free graphs to $\mathcal{O}(|V||E|)$, which constitutes a bottleneck for the complexity of their algorithm for the MWSSP in claw-free graphs. In this paper we give a new structural characterization of {claw, net}-free graphs with stability number not smaller than four which allows us to define a $\mathcal{O}(|V|\sqrt{|E|})$ time algorithm for the MWSSP in such graphs. This result, together with the $\mathcal{O}(|E| \log |V|)$ time algorithm for the MWSSP in claw-free graphs with stability number at most three described in [7], provides a $\mathcal{O}(\sqrt{|E|}(|V| + \sqrt{|E|} \log |V|))$ time algorithm for the MWSSP in {claw, net}-free graphs which improves the result of [6].

We say that a node $v \in V$ is *regular* if its neighborhood can be partitioned into two cliques. A maximal clique Q is *reducible* if $\alpha(N(Q)) \le 2$. If Q is a maximal clique, two non-adjacent nodes $u, v \in N(Q)$ are said to be Q-*distant* if $N(u) \cap N(v) \cap Q = \emptyset$ and Q-*close* otherwise ($N(u) \cap N(v) \cap Q \neq \emptyset$). A maximal clique Q is *normal* if it has three independent neighbors that are mutually Q-distant. In [8] Lovász and Plummer proved the following useful properties of a maximal clique in a claw-free graph.

Proposition 1.1. *Let $G(V, E)$ be a claw-free graph. If Q is a maximal clique in G then:*

(i) *if u and v are Q-close nodes then $Q \subseteq N(u) \cup N(v)$;*
(ii) *if u, v, w are mutually non-adjacent nodes in $N(Q)$ and two of them are Q-distant then any two of them are Q-distant and hence Q is normal.* □

Observe that a {claw, net}-free graph does not contain normal cliques.

Theorem 1.1. *Let $G(V, E)$ be a {claw, net}-free graph and u a regular node in V whose closed neighborhood is covered by two maximal cliques Q and $\overline{Q}$. Then Q ($\overline{Q}$) is reducible.*

Proof. Suppose, by contradiction, that $\alpha(N(Q)) \ge 3$. Let v_1, v_2, v_3 be three mutually non-adjacent nodes in $N(Q)$. If u is not adjacent to v_1, v_2, v_3, then by (i) of Proposition 1.1 we have that v_1, v_2, v_3 are mutually Q-distant and hence Q is normal, a contradiction. Consequently, without loss of generality, we can assume that u is adjacent to v_1 and so v_1 belongs to $\overline{Q}$. The nodes v_2 and v_3 do not belong to $Q \cup \overline{Q}$ and hence are not adjacent to u. It follows, again by (i) of Proposition 1.1, that v_2 and v_3 are distant with respect to Q. But then, by (ii) of Proposition 1.1, Q is a normal clique, a contradiction. □

Let G be a claw-free graph and let S be a stable set of $G(V, E)$. Any node $s \in S$ is said to be *stable*; any node $v \in V \setminus S$ satisfies $|N(v) \cap S| \le 2$ and is called *superfree* if $|N(v) \cap S| = 0$, *free* if $|N(v) \cap S| = 1$ and *bound* if $|N(v) \cap S| = 2$. For each free node u we denote by $S(u)$ the unique node in S adjacent to u.