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On the separation of split inequalities for non-convex quadratic integer programming ☆

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ABSTRACT

We investigate the computational potential of split inequalities for non-convex quadratic integer programming, first introduced by Letchford (2010) and further examined by Burer and Letchford (2012). These inequalities can be separated by solving convex quadratic integer minimization problems. For small instances with box-constraints, we show that the resulting dual bounds are very tight; they can close a large percentage of the gap left open by both the RLT- and the SDP-relaxations of the problem. The gap can be further decreased by separating the so-called non-standard split inequalities, which we examine in the case of ternary variables.

1. Introduction

The standard formulation of an (unconstrained) Integer Quadratic Programming Problem (IQP) is the following:

$$\min\{x^\top Qx + L^\top x + c \mid x \in \mathbb{Z}^n,\ l \le x \le u\} \tag{1}$$

with $Q \in \mathbb{Q}^{n\times n}$, $L \in \mathbb{Q}^n$, $c \in \mathbb{Q}$, $l \in (\mathbb{Z} \cup \{-\infty\})^n$, and $u \in (\mathbb{Z} \cup \{\infty\})^n$. We assume Q to be symmetric without loss of generality. However, we do not require Q to be positive semidefinite. In other words, we do not assume convexity of the objective function

$$f(x) := x^\top Qx + L^\top x + c.$$

Problem (1) is thus NP-hard both by the non-convexity of the objective function and by the integrality constraints on the variables. More precisely, the problem remains NP-hard in the convex case, i.e., when $Q \succeq 0$, even if all bounds are infinite or if all variables are binary. In the first case, Problem (1) is equivalent to the *closest vector problem* [1]; in the second case, it is equivalent to *binary quadratic programming* and *max-cut* [2]. Moreover, if f is non-convex and integrality is relaxed, i.e., if the variable x_i can be chosen in the interval $[l_i, u_i]$, the resulting problem is called *BoxQP* and is again NP-hard.

One approach for solving (1) is based on the idea of getting rid of the non convexity of f and then using a convex IQP solver. Billionnet et al. [3–5] proposed an approach consisting in reformulating the objective function and obtaining an equivalent one with a convex quadratic objective function. The approach aims at finding a convex reformulation that gives

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the highest value of its continuous relaxation. However, convexification requires a binary expansion of each non-binary variable, resulting in a large number of additional variables and possibly leading to numerical problems.

Another natural approach to get rid of the non-convexity of f consists in linearization. This approach has been investigated intensively in the literature. Let S_k be the set of symmetric matrices of dimension k. Using the linearization function $\ell\colon \mathbb{Q}^n \to S_{n+1}$ defined by

$$\ell(x) = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^\top$$

and setting

$$\tilde{Q} = \begin{pmatrix} c & \frac{1}{2}L^\top \\ \frac{1}{2}L & Q \end{pmatrix}$$

we can replace Problem (1) by the following equivalent problem:

$$\begin{array}{ll} \min & \langle \tilde{Q}, \ell(x) \rangle \\ \text{s.t.} & x \in \mathbb{Z}^n \\ & x_i \in [l_i, u_i] \quad \text{for } i = 1, \dots, n \end{array} \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

We can hence work in an extended space introducing a new set of variables X_{ij} with $i = 0, \dots, n$. The new linearized formulation is obtained by substituting each product $x_i x_j$, appearing in row i and column j of $\ell(x)$, by a new variable X_{ij}. For consistency, in this new formulation the linear component x_i is substituted by the new variable X_{0i}. Finally, for a reason that will later become clearer, we also introduce a new variable X_{00}. In this way the new space contains the original n variables and $1 + \binom{n}{2}$ new variables. All variables are collected in a symmetric matrix X of dimension $n + 1$, representing $\ell(x)$. The dimension of the extended space is thus

$$d(n) = \frac{(n+1)(n+2)}{2} - 1.$$

The main challenge is now to ensure $X = \ell(x)$. This is equivalent to requiring

$$X_{00} = 1, \qquad \text{rank}(X) = 1, \quad \text{and} \quad X \succeq 0.$$

Hence, one way to reformulate Problem (1) is as follows:

$$\begin{array}{ll} \min & \langle \tilde{Q}, X \rangle \\ \text{s.t.} & X_{00} = 1 \\ & \text{rank}(X) = 1 \\ & X \succeq 0 \\ & X_{i0} \in \mathbb{Z} \quad \text{for } i = 1, \dots, n \\ & X_{i0} \in [l_i, u_i] \quad \text{for } i = 1, \dots, n. \end{array} \tag{3}$$

Working in the X-space allows more freedom and several reformulations and relaxations of Problem (3) can be defined. By eliminating the rank constraint and the integrality constraints we obtain the SDP-relaxation (SDP) of Problem (3). Buchheim and Wiegele [6] devise a branch-and-bound algorithm based on this continuous relaxation.

An alternative is to work with an ILP formulation and then use its continuous relaxation for computing bounds. Sherali and Adams [7] proposed a unifying framework for strengthening the linearization using the so-called RLT inequalities. By taking into account the bounds on the original variables $l_i \le x_i \le u_i$, we can add the following RLT-inequalities:

$$\begin{array}{l} X_{ij} - l_i X_{j0} - l_j X_{i0} \ge -l_i l_j \\ X_{ij} - u_i X_{j0} - u_j X_{i0} \ge -u_i u_j \\ -X_{ij} + l_i X_{j0} + u_j X_{i0} \ge l_i u_j \\ -X_{ij} + u_i X_{j0} + l_j X_{i0} \ge u_i l_j. \end{array}$$

The four inequalities above were originally introduced by McCormick [8]. Anstreicher [9] uses RLT-inequalities for strengthening SDP-relaxations for BoxQP problems. He investigates the relation between the SDP-relaxation, the linear relaxation with RLT inequalities and the SDP-relaxation with RLT inequalities. He shows that adding RLT inequalities to the SDP-relaxation improves the resulting bounds. From a practical point of view, a drawback of this approach is that SDP solvers have problems in handling the additional RLT-inequalities.

Another class of valid inequalities that can be used for strengthening the extended formulation are the so-called psd inequalities, introduced by Laurent and Poljak [10]:

$$\langle vv^\top, X \rangle \ge 0 \quad \forall v \in \mathbb{Q}^{n+1}. \tag{4}$$