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Symmetry-exploiting cuts for a class of mixed-0/1 second-order cone programs[☆]

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ABSTRACT

We will analyze mixed-0/1 second-order cone programs where the continuous and binary variables are solely coupled via the conic constraints. We devise a cutting-plane framework based on an implicit Sherali–Adams reformulation. The resulting cuts are very effective as symmetric solutions are automatically cut off and each equivalence class of 0/1 solutions is visited at most once. Further, we present computational results showing the effectiveness of our method and briefly sketch an application in optimal pooling of securities.

1. Introduction

Mixed-integer second-order cone programming, as one of the rather tractable problem classes of mixed-integer nonlinear programming, gained strong interest in recent years. Due to their structure and convexity, the underlying relaxations can be solved in polynomial time to ϵ-optimality. Mixed-integer second-order cone programs are usually solved via outer approximation algorithms or within a branch-and-cut framework (see e.g., [1,2] or [3] for an overview).

Here we consider a restricted class of mixed-integer second-order cone programs where the integral variables are binary and the only coupling of binary and continuous variables occurs in the conic constraints. For this problem class, we derive cutting-planes based on an implicit Sherali–Adams reformulation of the problem and subsequent application of subgradient based cuts (see e.g., [3–5]).

The Sherali–Adams hierarchy (cf. [6]) is a well-known tool to generate successively tighter linear approximations of the integral hull of a polytope. Recently, it regained strong interest, in particular in connection with inapproximability of certain combinatorial problems as well as in the context of algebraic geometry (see e.g., [7–10]). We will use the Sherali–Adams hierarchy on an implicit level here by interpreting the second-order cone program as the intermediate step of the Sherali–Adams procedure. In contrast to the Sherali–Adams hierarchy, which is used for convexification, outer approximation algorithms (see [4,11–13]) are used to decompose a problem into a master problem and subproblems. Given

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an optimal solution to the master problem, one solves the subproblem and derives cutting planes which in turn are added back to the master problem and the process repeats.

The first part of our reformulation exploits the fact that for a binary variable x it holds $x^2 - x = 0$ in order to strengthen the formulation. Applying this reformulation technique, we disentangle the binary variables from the continuous variables in the conic constraints. As a result we obtain a reformulation where the binary variables are aggregated in the linear parts of the conic conditions. In a second step we derive a family of cutting planes based on subgradients of the constraint functions which will be used in an outer approximation framework. While generalized Benders cut and classical outer approximation are often outperformed by more advanced methods, the proposed combination works very well in this case. In fact, their effectiveness stems from their ability to cut off equivalence classes of symmetric solutions, thus vastly reducing the number of outer approximation iterations. We apply these cuts and the resulting decomposition to solve a pooling problem arising in portfolio optimization in finance.

Related work. Cutting-planes for mixed-integer second-order cone programs have been extensively investigated. For example in [14] lift-and-project based cuts for mixed 0/1 conic programming problems have been studied. In [15], Gomory mixed-integer rounding cuts for second-order cone programs have been devised and in [16] lifting for conic mixed-integer programming was investigated. A branch-and-cut based method for convex mixed 0/1 programming was outlined in [17] and in [18] lift-and-project based cutting-planes as well as subgradient based outer approximations have been applied to solve mixed-integer second-order cone programs. A lifted linear programming branch-and-bound algorithm for second-order cone programs, where second-order cone constraints are approximated via linear ones, was outlined in [1]. Our approach is different as we consider a special class of mixed 0/1 second-order cone programs whose structure we exploit.

Our contribution. We propose a reformulation technique and resulting cutting planes for a special class of second-order cone programs where the coupling of continuous and binary variables occurs only in the conic constraints. In a first step the formulation is strengthened by applying the Sherali–Adams closure in an implicit fashion. For this strengthened reformulation we derive strong cutting planes, which are applied in an outer approximation framework and evaluated in computational experiments.

Outline. In Section 2 we briefly recall the necessary notation, followed by the reformulation of the considered problem class in Section 3. We then introduce the cutting-plane framework and derive cutting-planes in Section 4 based on this reformulation. We conclude with a pooling problem arising in portfolio optimization in Section 5, computational results in Section 6, and some final remarks in Section 7.

2. Preliminaries

For $n \in \mathbb{N}$ we write $[n] := \{1, \dots, n\}$ and for a set $J \subseteq [n]$, we define $\bar{J} := [n] \setminus J$ to be the *complement* of J in $[n]$. All other notation is standard as to be found in [18,19].

We will consider the following class of second-order cone programs:

Definition 2.1. A *weakly-coupled* 0/1 *second-order cone program* (wSOC) has the form:

$$\min\ c_J x_J + c_{\bar{J}} x_{\bar{J}} \tag{wSOC}$$

$$\text{s.t. } A_J x_J \le b_J \tag{2.1}$$

$$A_{\bar{J}} x_{\bar{J}} \le b_{\bar{J}} \tag{2.2}$$

$$\left\| (W_\ell x_{J_\ell}, Q_\ell x_{\bar{J}_\ell}) \right\|_2 \le \alpha_\ell x_\ell^0 + \tau_\ell \quad \forall\ \ell \in [k] \tag{2.3}$$

$$x_{\bar{J}} \in \mathcal{K} \tag{2.4}$$

$$x_{\bar{J}} \in \mathbb{R}^{|\bar{J}|} \tag{2.5}$$

$$x_J \in \{0,1\}^{|J|} \tag{2.6}$$

$$x_\ell^0 \in \{0,1\} \quad \forall\ \ell \in [k], \tag{2.7}$$

with $n \in \mathbb{N}, J \subseteq [n], k \in \mathbb{N}, A_J \in \mathbb{R}^{m_J \times |J|}, W_\ell \in \mathbb{R}^{m_{J_\ell} \times |J_\ell|}, A_{\bar{J}} \in \mathbb{R}^{m_{\bar{J}} \times |\bar{J}|}, Q_\ell \in \mathbb{R}^{m_{\bar{J}_\ell} \times |\bar{J}_\ell|}, b_J \in \mathbb{R}^{m_J}, b_{\bar{J}} \in \mathbb{R}^{m_{\bar{J}}}, c = (c_J, c_{\bar{J}}) \in \mathbb{R}^n$, and $\alpha_\ell, \tau_\ell \in \mathbb{R}$ for all $\ell \in [k]$. Further let $J_\ell \subseteq J$ and $\bar{J}_\ell \subseteq \bar{J}$ for all $\ell \in [k]$. The index set J_ℓ denotes the binary variables of the ℓth second-order cone constraint and similarly $\bar{J}_\ell$ denotes the continuous variables. Moreover, $\mathcal{K} \subseteq \mathbb{R}^{|\bar{J}|}$ in Condition (2.4) denotes a convex feasible region arising from additional second-order cone constraints on the continuous variables $x_{\bar{J}}$.

Note that additional second-order cone constraints solely in the binary variables can be rewritten as linear ones with the same feasible solutions using standard techniques and thus can be considered to be included in (2.1); appearing mixed terms can be linearized, e.g., with McCormick linearizations. The second-order cone conditions (2.3) resemble a typical on/off constraint where the leading cone variable is given by an affine-linear function in a 0/1 variable. We slightly abuse notation by using $\bar{J}_\ell$ to refer to the index subset of continuous variables contained in the ℓth second-order cone constraint, i.e., we