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The quadratic balanced optimization problem ☆

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HIGHLIGHTS

- Introduce the quadratic balanced optimization problem.
- Complexity results, algorithms, and polynomially solvable special cases.
- Experimental analysis of algorithms.

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ABSTRACT

We introduce the quadratic balanced optimization problem (QBOP) which can be used to model equitable distribution of resources with pairwise interaction. QBOP is strongly NP-hard even if the family of feasible solutions has a very simple structure. Several general purpose exact and heuristic algorithms are presented. Results of extensive computational experiments are reported using randomly generated quadratic knapsack problems as the test bed. These results illustrate the efficacy of our exact and heuristic algorithms. We also show that when the cost matrix is specially structured, QBOP can be solved as a sequence of linear balanced optimization problems. As a consequence, we have several polynomially solvable cases of QBOP.

1. Introduction

Let $E = \{1, 2, \dots, m\}$ be a finite set and $\mathcal{F}$ be a family of non-empty subsets of E. It is assumed that $\mathcal{F}$ is represented in a compact form of size polynomial in m without explicitly listing its elements. For each $(i, j) \in E \times E$, a cost c_{ij} is prescribed. The elements of $\mathcal{F}$ are called feasible solutions and the $m \times m$ matrix $C = (c_{ij})$ is called the *cost matrix*. Then the *quadratic balanced optimization problem* (QBOP) is to find $S \in \mathcal{F}$ such that

$$\max\{c_{ij} : (i, j) \in S \times S\} - \min\{c_{ij} : (i, j) \in S \times S\}$$

is as small as possible.

QBOP is closely related to the *balanced optimization problem* introduced by Martello et al. [1]. To emphasize the difference between the balanced optimization problem of [1] and QBOP, we call the former a *linear balanced optimization problem*

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(LBOP). Special cases of LBOP were studied by many authors [1–10]. Optimization problems with objective functions similar to that of LBOP have been studied by Zeitlin [11] for resource allocation, by Gupta and Sen [12], Liao and Huang [13], and Tegze and Vlach [14–16] for machine scheduling, by Ahuja [17] for linear programming, by Scutellà [18] for network flows, and by Liang et al. [19] for workload balancing. A generalization of LBOP where elements of E are categorized has been studied by Berežný and Lacko [20,21] and Grinèová, Kravecová, and Kuláè [22]. Punnen and Nair [23] studied LBOP with an additional linear constraint. Punnen and Aneja [24] and Turner et al. [25] studied the lexicographic version of LBOP. To the best of our knowledge, QBOP has not been studied in literature so far.

Most of the applications of LBOP discussed in literature translate into applications of QBOP by interpreting c_{ij} as the pairwise interaction weight of elements i and j in E. To illustrate this, let us consider the following variation of the travel agency example of Martello et al. [1]. A North American travel agency is planning to prepare a European tour package. Its clients travel from New York to London by a chartered flight. The clients have the option to choose a maximum of two tourist locations from an available set S of locations. If a client chooses locations i and j, then c_{ij} is the total tour time. There are m potential locations and the company wishes to choose $k = |S|$ locations to be included in the package so that the duration of tours for any pair of locations is approximately the same. This way, one can avoid waiting time of clients in London, after their tour and the whole group can return by the same chartered flight. The objective of the tour company can be represented as Minimizing $\max\{c_{ij} : (i, j) \in S \times S\} - \min\{c_{ij} : (i, j) \in S \times S\}$ while satisfying appropriate constraints.

Other applications of the model include balanced portfolio selection for managing investment accounts where risk estimates on pairs of investment opportunities are to be considered because of hedging positions and participant selection for psychological experiments where it is important that all the people in the group equally know each other.

The objective function of QBOP can be viewed as range of a covariance matrix C associated with a combinatorial optimization problem. In this case, BQOP attempts to minimize a dispersion measure. Minimization of various measures of dispersion such as variance, absolute deviation from the mean etc. has been studied in the context of combinatorial optimization problems [26,27]. However, none of these studies take into consideration information from the covariance matrix which measures impact of pairwise interaction. This interpretation leads to other potential applications of our model.

In this paper we study QBOP and propose several general purpose algorithms. The polynomial solvability of these algorithms are closely related to that of an associated feasibility problem. QBOP is observed to be NP-hard even if the family $\mathcal{F}$ of feasible solutions has very simple structure. We also investigate QBOP when the cost matrix C has a decomposable structure, i.e., $c_{ij} = a_i + b_j$ or $c_{ij} = a_i b_j$. In each of these special cases, we show that QBOP can be solved in polynomial time whenever the corresponding LBOP can be solved in polynomial time. As a consequence, we have $O(m^2 \log n)$ and $O(n^6)$ algorithms for QBOP when $\mathcal{F}$ is chosen as spanning trees of a graph on n nodes and m edges or perfect matchings on a $K_{n,n}$, respectively. Our general purpose exact algorithms can be modified into heuristic algorithms. Some sufficient conditions are derived to speed up these algorithms and their effect is analyzed using extensive experimentation in the context of quadratic balanced knapsack problems. We also compared the heuristic solutions with exact solutions and the results establish the efficiency of our heuristic algorithms.

The paper is organized as follows. In Section 2 we discuss the complexity of the problem and introduce notations and definitions. Section 3 deals with exact and heuristic algorithms. In Section 4 we present our polynomially solvable special cases. In Section 5 we discuss the special case of the quadratic balanced knapsack problem. Experimental results are presented in Section 6. In Section 7, a generalization QBOP where interaction between k-elements are considered instead of two elements as in the case of QBOP. Concluding remarks are presented in Section 8.

2. Complexity and notations

Without loss of generality, we assume that $c_{ij} \geq 0$ for otherwise we can add a large constant to all c_{ij} values to get an equivalent problem with non-negative cost values. It may be noted that when $c_{ij} \geq 0$ for all $(i, j) \in E \times E$ and $c_{ii} = 0$ for $i \in E$, QBOP reduces to the *quadratic bottleneck problem* (QBP) [28,29]. QBP is NP-hard even if $\mathcal{F}$ is the collection of all subsets of E with cardinality no more than k for a given k, which depends on m [29]. In fact, for such a problem, computing an ϵ-optimal solution is also NP-hard for any $\epsilon > 0$ even if $c_{ij} \in \{0, 1\}$ and $c_{ii} = 0$ [29]. As an immediate consequence, it can be verified that for the corresponding instance of QBOP, computing an ϵ-optimal solution is NP-hard for any $\epsilon > 0$. In contrast, the corresponding LBOP is polynomially solvable. Thus, the complexity of QBOP and LBOP are very different and QBOP apparently is a more difficult problem.

For a given cost matrix C and $S \in \mathcal{F}$, we denote

$$Z_{\max}(C, S) = \max\{c_{ij} : (i, j) \in S \times S\},$$
$$Z_{\min}(C, S) = \min\{c_{ij} : (i, j) \in S \times S\} \quad \text{and}$$
$$Z(C, S) = Z_{\max}(C, S) - Z_{\min}(C, S).$$

For a given family of feasible solutions, we use the notation QBOP(C) to indicate that the cost matrix under consideration for QBOP is C. Thus, QBOP(C) and QBOP(C^*), where $C \neq C^*$, are two instances of QBOP with the same family of feasible solutions but different cost matrices C and C^* respectively.

For any two real numbers α and β such that $\alpha \leq \beta$ and cost matrix C, let $F(C, \alpha, \beta) = \{S \in \mathcal{F} : Z_{\min}(C, S) \geq \alpha$ and $Z_{\max}(C, S) \leq \beta\}$ and $E(C, \alpha, \beta) = \{(i, j) : c_{ij} < \alpha$ or $c_{ij} > \beta\}$. Then the *quadratic feasibility problem* can be stated as follows: