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Geometric versions of the three-dimensional assignment problem under general norms

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ABSTRACT

We discuss the computational complexity of special cases of the three-dimensional (axial) assignment problem where the elements are points in a Cartesian space and where the cost coefficients are the perimeters of the corresponding triangles measured according to a certain norm. (All our results also carry over to the corresponding special cases of the three-dimensional matching problem.)

The minimization version is NP-hard for every norm, even if the underlying Cartesian space is 2-dimensional. The maximization version is polynomially solvable, if the dimension of the Cartesian space is fixed and if the considered norm has a polyhedral unit ball. If the dimension of the Cartesian space is part of the input, the maximization version is NP-hard for every L_p norm; in particular the problem is NP-hard for the Manhattan norm L_1 and the Maximum norm L_∞ which both have polyhedral unit balls.

1. Introduction

The three-dimensional (axial) assignment problem (3AP) is an important and well-studied problem in combinatorial optimization. An instance of the 3AP consists of three sets X, Y, Z with $|X| = |Y| = |Z| = n$, and a cost function $c : X \times Y \times Z \to \mathbb{R}$. The goal is to find a set of n triples in $X \times Y \times Z$ that cover every element in $X \cup Y \cup Z$ exactly once, such that the sum of the costs of these triples is minimized. In the closely related maximization version max-3AP of the 3AP, this sum is to be maximized. The book [1] by Burkard, Dell'Amico and Martello contains a wealth of information on the 3AP and other assignment problems.

A prominent special case of the 3AP is centered around some metric space (S, d) where S is a set and where d is a distance function on $S \times S$ (that hence is symmetric, non-negative, and satisfies the triangle inequality).

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The elements in $X \cup Y \cup Z$ are points in S, and the cost $c(x, y, z)$ of a triple $(x, y, z) \in X \times Y \times Z$ is given by

$$c(x, y, z) = d(x, y) + d(y, z) + d(z, x). \tag{1}$$

Costs of this type are called *perimeter costs*; intuitively speaking, they measure the perimeter of the triangle determined by three points x, y, z in the metric space.

The 3AP is well-known to be NP-hard; see for instance Karp [2] or Garey and Johnson [3]. Spieksma and Woeginger [4] establish NP-hardness of the special case of perimeter costs (1) where the underlying metric space is the two-dimensional Euclidean plane with standard Euclidean distances. Crama and Spieksma [5] design a polynomial time approximation algorithm with worst case guarantee 4/3 for the 3AP with perimeter costs; their approach works for arbitrary metric spaces without imposing any additional structural constraints. Burkard, Rudolf and Woeginger [6] exhibit a polynomially solvable special case of the max-3AP where the costs are decomposable and products of certain parameters.

Results of this paper. We study 3AP and max-3AP with perimeter costs in Cartesian spaces under arbitrary distance functions. On the negative side, we derive NP-hardness results that contain and generalize the known results from the literature for the standard Euclidean distances. On the positive side, we derive polynomial time algorithms for certain special cases of max-3AP where the distances are defined via norms with polyhedral unit balls. Our main results are the following:

(A) Problem max-3AP is polynomially solvable, if the dimension of the underlying Cartesian space is a fixed constant and if the underlying norm has a polyhedral unit ball.

(B) Problem max-3AP is NP-hard, if the dimension of the underlying Cartesian space is part of the input and if the underlying norm is any fixed L_p norm. This hardness result in particular holds for the Manhattan norm L_1 and the Maximum norm L_∞ which both have polyhedral unit balls.

(C) Finally, the minimization problem 3AP is NP-hard for any fixed norm, even if the underlying Cartesian space is 2-dimensional.

Result (A) heavily builds on the machinery developed by Barvinok, Fekete, Johnson, Tamir, Woeginger and Woodroofe [7] for the Travelling Salesman Problem (TSP). Also the TSP is polynomially solvable, if the cities are points in some Cartesian space of fixed dimension and if the distances are defined via norms with polyhedral unit balls. While the framework for our result (A) is taken from [7], the technical details and the combinatorial features are very different and require a number of new ideas. Result (B) is done by a routine NP-hardness reduction from a closely related NP-hard graph problem. Result (C) builds on the NP-hardness reductions of Spieksma and Woeginger [4] and Pferschy, Rudolf and Woeginger [8] for Euclidean distances. In the Euclidean case, one may use Pythagorean triangles as simple building blocks to control the distances between points and to ensure rational coordinates that can be processed by a Turing machine. In the general case (C), it is much more tedious to prove the existence of the corresponding building blocks.

Organization of this paper. Section 2 summarizes some standard geometric definitions around distances, norms and unit balls. Result (A) for the max-3AP is derived in two steps. First Section 3 derives an auxiliary result on the max-3AP under so-called tunneling distances, and then Section 4 establishes that max-3AP under polyhedral norms is a special case of the tunneling case. Section 5 contains the proof of result (B). Section 6 constructs certain lattices with certain useful properties; these lattices are then used in Section 7 to prove the NP-hardness result (C). Section 8 translates our results (A), (B) and (C) into corresponding results for the maximization version and the minimization version of the three-dimensional matching problem. Finally, Section 9 concludes the paper with a short discussion and some open questions.