Discrete Optimization 18 (2015) 189–192

Contents lists available at ScienceDirect

Discrete Optimization

www.elsevier.com/locate/disopt

Note

Intersection cuts—standard versus restricted

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ARTICLE INFO

Article history:
Received 29 July 2015
Received in revised form 8 October 2015
Accepted 8 October 2015
Available online 4 November 2015

Keywords:
MIP
Cut
Intersection

ABSTRACT

This note is meant to elucidate the difference between intersection cuts as originally defined, and intersection cuts as defined in the more recent literature. It also states a basic property of intersection cuts under their original definition.

Intersection cuts for mixed integer programs were introduced in the early 1970s [1,2] as inequalities obtained by intersecting the extreme rays of the polyhedral cone $C(B)$, where B is a basis of the linear programming relaxation P, with the boundary of some convex set T whose interior contains the vertex $v(B)$ of P but no *feasible integer point*. Such a set T will be called P_I*-free*, where P_I is the set of feasible integer points.

In particular, if the simplex tableau associated with the basis B is

$$x_B = \bar{x}_B - \sum_{j \in J} \bar{a}_j x_j,$$

where J indexes the co-basis of B (i.e. the set of nonbasic variables) and if the extreme rays

$$\begin{pmatrix} \bar{x}_B \\ 0 \end{pmatrix} + \begin{pmatrix} -\bar{a}_j \\ e_j \end{pmatrix} \lambda_j, \quad j \in J$$

of the cone $C(B)$ (where e_j is the jth unit vector) intersect the boundary of T at the points defined by $\lambda_j = \lambda_j^*$, $j \in J$, then the hyperplane through these n points defines the intersection cut

$$\sum_{j \in J} \frac{1}{\lambda_j^*} x_j \geq 1. \tag{1}$$

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http://dx.doi.org/10.1016/j.disopt.2015.10.001
1572-5286/

More recently, intersection cuts became the focus of renewed interest as a result of the seminal paper by Andersen, Louveaux, Weismantel and Wolsey [3], which highlights their significance in the context of cut generation from multiple rows of the simplex tableau. However, this paper and the ensuing voluminous literature used a narrower definition of intersection cuts, namely as inequalities obtained by intersecting the extreme rays of $C(B)$ with the boundary of some convex set T' whose interior contains $v(B)$ but *no integer point*. Such a set T' is called *lattice-free*. This definition is more restrictive than the original one, since it excludes intersection cuts obtained from P_I-free sets that are not lattice-free, whereas the original definition includes all intersection cuts from convex lattice-free sets, as these are all P_I-free. In the sequel we will refer to intersection cuts obtained from P_I-free convex sets as standard (SIC), and to those obtained from lattice-free convex sets as restricted (RIC).

In most of the specific cases considered so far in the literature this difference does not matter, since the lattice-free sets used to generate cuts are P_I-free. This is the case with split cuts and cuts obtained by combining splits, like cuts from triangles or quadrilaterals. But intersection cuts generated from P_I-free convex sets that are *not* lattice-free can be much stronger than those generated from lattice-free sets. For instance, if the lattice-free set T' has a facet whose relative interior contains only infeasible integer points, then switching to a P_I-free set T larger than T' may yield a stronger cut. Furthermore, intersection cuts from a lattice-free set T', when expressed in terms of the nonbasic variables, have all their coefficients nonnegative, as is easily seen from the definition (1) of the cut. On the other hand, intersection cuts from a P_I-free set may have negative coefficients in terms of the nonbasic variables. This is easiest to see if we express the intersection cut from the P_I-free polyhedron T with facets defined by $\sum_{j\in J} d_{ij}x_j \le d_{i0}$, $i \in Q$, as disjunctive cuts, $\delta x \ge 1$ from $\vee_{i\in Q}(\sum_{j\in J} d_{ij}x_j \ge d_{i0})$, having coefficients

$$\delta_j = \max_{i\in Q} \frac{d_{ij}}{d_{i0}}, \quad j \in J.$$

Clearly, if $d_{ij} < 0$ for all $i \in Q$, then $\delta_j < 0$. This cannot occur for a lattice-free convex set T', since in the case of the latter, the only rays that do no intersect the boundary of T' are those parallel to some facet of T', in which case they have $d_{ij} = 0$ in the inequality defining that facet.

Example. Consider the instance

$$\begin{aligned} \min\ & x_1 + 2x_2 \\ & 4x_1 + 4x_2 \ge 3 \\ & -x_1 + 3x_2 \ge \frac{5}{4} \\ & 2x_1 + 4x_2 \le 5 \\ & x_1, x_2 \ge 0 \quad \text{integer} \end{aligned}$$

whose linear programming relaxation is the shaded area in Fig. 1. The optimal LP solution is $\bar{x} = (\frac{1}{4}, \frac{2}{4})$, and the associated simplex tableau is

		x_1	x_2	s_1	s_2	s_3
x_1	$\frac{1}{4}$	1		$\frac{3}{16}$	$-\frac{1}{4}$	
x_2	$\frac{1}{2}$		1	$\frac{1}{16}$	$\frac{1}{4}$	
s_3	$\frac{5}{2}$			$-\frac{5}{8}$	$-\frac{1}{2}$	1

The intersection cut from the lattice-free triangle $T' := \{x \in \mathbb{R}_2 : x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 2\}$, shown in Fig. 2, is $(-\frac{1}{19})x_1 + x_2 \ge \frac{3}{4}$, defined by the two intersection points $\left(0, \frac{3}{4}\right)$ and $\left(\frac{19}{16}, \frac{13}{16}\right)$.