Discrete Optimization 8 (2011) 174–190

Contents lists available at ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

An exact approach for the Vertex Coloring Problem

Enrico Malaguti [a,*], Michele Monaci [b], Paolo Toth [a]

[a] Dipartimento di Elettronica, Informatica e Sistemistica, University of Bologna, Viale Risorgimento, 2-40136, Bologna, Italy

[b] Dipartimento Ingegneria dell'Informazione, University of Padova, Viale Gradenigo, 6/A-35131, Padova, Italy

ARTICLE INFO

Article history:
Received 18 December 2009
Received in revised form 27 July 2010
Accepted 31 July 2010
Available online 21 August 2010

Keywords:
Vertex Coloring
Column generation
Branch-and-Price
Computational experiments

ABSTRACT

Given an undirected graph $G = (V, E)$, the *Vertex Coloring Problem* (VCP) requires to assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized. In this paper, we present an exact algorithm for the solution of VCP based on the well-known Set Covering formulation of the problem. We propose a Branch-and-Price algorithm embedding an effective heuristic from the literature and some methods for the solution of the slave problem, as well as two alternative branching schemes. Computational experiments on instances from the literature show the effectiveness of the algorithm, which is able to solve, for the first time to proven optimality, five of the benchmark instances in the literature, and reduce the optimality gap of many others.

1. Introduction

Given an undirected graph $G = (V, E)$, the *Vertex Coloring Problem* (VCP) requires to assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized.

The Vertex Coloring Problem is one of the classical NP-hard problems (see [1]), and it is well known not only for its theoretical aspects and for its difficulty from the computational point of view, but also because it appears in many real world applications, including, among many others, scheduling [2,3], timetabling [4], register allocation [5], frequency assignment [6] and communication networks [7].

Despite the relevance of the problem, few exact approaches have been proposed in the literature, and they are able to consistently solve only small size instances. The only recent contribution presenting extensive computational results on benchmark instances is based on the so-called descriptive formulation, which is strengthened by means of inequalities derived from the structure of the problem (see [8,9]). In this paper we present an effective algorithm for the exact solution of VCP, based on the alternative Set Covering formulation, for which the last paper presenting extensive computational results on commonly considered benchmark instances dates back to 1996 (see [10]). For speeding up the solution of the continuous relaxation of the Set Covering formulation, we propose effective methods, including the use of a metaheuristic procedure and of the advanced capabilities of modern *Integer Linear Programming* (ILP) solvers. In addition, we compare two alternative branching schemes. Finally, the use of an effective metaheuristic algorithm for VCP, used to initially produce a very good upper bound, determines a further improvement of the effectiveness of the algorithm.

The perspective of this paper is mainly computational, and we think that its main contributions are:

- presenting an effective exact algorithm for VCP, able to solve, for the first time to proven optimality, some instances which have been "open" for several years, and to improve the best lower bound of many others;

* Corresponding author. Tel.: +39 0512093942; fax: +39 0512093073.
E-mail addresses: enrico.malaguti@unibo.it (E. Malaguti), monaci@dei.unipd.it (M. Monaci), paolo.toth@unibo.it (P. Toth).

1572-5286/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.disopt.2010.07.005

- illustrating an effective integration of metaheuristic approaches and exact methods;
- showing how modern computational tools can be used to tackle a branching scheme which otherwise would be hard to implement.

We recall here some definitions used in the following. Let n and m be the cardinalities of vertex set V and edge set E, respectively. A subset of V is called a *stable set* if no two adjacent vertices belong to it (note that, in VCP, all the vertices having the same color form a stable set, and viceversa). A *clique* of a graph G is a complete subgraph of G (note that the size of a clique represents a valid lower bound for VCP). A stable set (resp. clique) is *maximal* if no vertex can be added still having a stable set (resp. clique). A *k coloring* of G is a partition of V into k stable sets. Each stable set of a coloring is called a *color class*. An optimal coloring of G is a *k coloring* with the smallest possible value of k (the *chromatic number* $\chi(G)$ of G). For each vertex $v \in V$, let $N(v)$ be the neighborhood of v, i.e., the set of vertices adjacent to v, and $\delta(v)$ the degree of v (i.e. $\delta(v) = |N(v)|$). We say that v *dominates* w if $N(w) \subset N(v)$.

The paper is organized as follows. In Section 1.1 we review the main contributions and models proposed in the literature for VCP, including the Set Covering formulation. In Section 2 we present our algorithm: Section 2.1 describes an efficient heuristic from the literature that is embedded in the exact method, Section 2.2 presents methods for effectively handling the exponential number of variables in the model. Two branching schemes are evaluated and discussed in Section 2.3. Finally, Section 3 gives the results of our experiments on a large set of instances from the literature, and Section 4 draws some conclusions.

1.1. Literature review

As mentioned in the introduction, VCP and its variants have been widely studied in the literature so far. We recall here the main contributions, and refer the reader to the recent survey by Malaguti and Toth [11] for an extensive discussion. We note that VCP appears either directly or as a subproblem in many real world applications in different contexts. However, the state-of-the-art exact algorithms for VCP are able to solve consistently only small randomly generated instances, with up to 100 vertices (see, e.g., [12]), whereas real world applications commonly deal with graphs having hundreds or thousands of vertices. This motivates the large amount of literature concerning the heuristic and metaheuristic approaches for VCP.

Among the greedy algorithms we mention the sequential algorithm (generally called SEQ), that considers the vertices in a given order and assigns each vertex to the lowest-indexed color class in which it fits, and the *Recursive Largest First* (RLF) algorithm by Leighton [2], which colors the vertices, one class at a time, in a greedy way. As to the metaheuristics, the first proposed algorithm was the Tabu Search procedure TABUCOL by Hertz and de Werra [13]. This algorithm solves the problem in its *decision form*, i.e., it receives in input a threshold value k representing the desired solution value, and considers k available colors, moving among complete colorings. Infeasible solutions, corresponding to infeasible color classes, are considered during the evolution of the algorithm and penalized in the objective function. Johnson et al. [14] proposed a simulated annealing algorithm and computationally studied different neighborhoods. Davis [15] presented a genetic algorithm in which each solution is encoded as a permutation of the vertices, which are then colored through the SEQ algorithm. The *Impasse Class Neighborhood* proposed by Morgenstern [16] turned out to be for a long time the most effective approach for VCP. Given a threshold value k, each solution S is a partition of the vertex set in $k+1$ color classes $\{V_1, \ldots, V_k, V_{k+1}\}$ in which all classes, but possibly the last one, are stable sets. Given the current solution, its neighborhood is defined as the set of solutions that can be obtained by moving a vertex from color class $k+1$ to a different color class, and is explored to minimize the global degree of the uncolored vertices:

$$f(S) = \sum_{v \in V_{k+1}} \delta(v). \tag{1}$$

An effective evolutionary algorithm HCA (Hybrid Coloring Algorithm) was proposed by Galinier and Hao [17]; this algorithm works with a fixed k, and combines an improved version of TABUCOL with a crossover operator which is specialized for the VCP, thus obtaining one of the most performing algorithms for the problem. Algorithm MIPS-CLR by Funabiki and Higashino [18] is a combination of a Tabu Search technique and of the *Impasse Class Neighborhood*; a major difference with most of the other algorithms is that MIPS-CLR works with a variable k, i.e. in the *optimization form* of VCP. Galinier et al. [19] proposed an *adaptive memory* algorithm AMACOL, that works with fixed k, and embeds an improved version of TABUCOL. Recently, Blöchliger and Zufferey [20] proposed two Tabu Search algorithms working with fixed k and based on the *Impasse Class Neighborhood*, while Plumettaz et al. [21] presented an Ant Colony scheme that incorporates a local search procedure adopting the same neighborhood.

Finally, we mention the evolutionary algorithm recently proposed by Malaguti et al. [22], based on the *Impasse Class Neighborhood* and a crossover operator, which is an adaptation of the *Greedy Partitioning Crossover* proposed by Galinier and Hao [17]. The evolutionary algorithm, which solves the problem for a fixed value of k, is then embedded in an overall algorithm, called MMT, which solves the optimization form of the problem and is described in Section 2.1.

The large amount of literature on the heuristics for VCP has not a similar counterpart for what concerns the exact methods. Among them, we mention the Branch-and-Bound algorithm proposed by Brown [23], which is based on the idea of coloring