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On the relative strength of different generalizations of split cuts

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ABSTRACT

Split cuts are among the most important and well-understood cuts for general mixed-integer programs. In this paper we consider some recent generalizations of split cuts and compare their relative strength. More precisely, we compare the elementary closures of split, cross, crooked cross and general multi-branch split cuts as well as cuts obtained from multi-row and basic relaxations.

We present a complete containment relationship between the closures of split, rank-2 split, cross, crooked cross and general multi-branch split cuts. More specifically, we show that 3-branch split cuts strictly dominate crooked cross cuts, which in turn strictly dominate cross cuts. We also show that multi-branch split cuts are incomparable to rank-2 split cuts. In addition, we also show that cross cuts, and hence crooked cross cuts, cannot always be obtained from 2-row relaxations or from basic relaxations. Together, these results settle some open questions raised in earlier papers.

1. Introduction

Cutting planes are crucial for solving mixed-integer programs (MIPs), and the Gomory mixed-integer (GMI) cut is currently among the most effective cutting planes for general MIPs. Cook, Kannan and Schrijver [1] studied a class of disjunctive cuts called *split cuts* (formal definitions are presented in Section 2). Despite their simplicity, many known families of cutting planes (or *cuts* for short) such as the GMI, lift-and-project, and flow cover cuts can be viewed as split cuts from very simple disjunctions. Due to their importance, these cuts have been extensively studied, both theoretically [2–8] and computationally [9–14].

In the following, we refer to a *mixed-integer set* as the set of mixed-integer solutions of a given finite set of rational linear equations or inequalities (a fixed subset of variables are restricted to be integral), and refer to the polyhedron defined by these linear constraints as the *linear relaxation* of the mixed-integer set. Cuts for a mixed-integer set are linear inequalities valid for its convex hull. Split cuts (and other structured families of cuts) for a mixed-integer set are assumed to be derived using both the linear constraints and the integrality restrictions defining the set. The *elementary closure* of a family of cuts for a mixed-integer set is the set of (real) points in its linear relaxation satisfying all cuts in the family.

When the linear constraints defining a mixed-integer set are given in inequality form, Andersen, Cornuéjols and Li [7] proved that any split cut can be obtained as a split cut from a *basic* relaxation; also see [15] for a simpler proof. A basic relaxation is the mixed-integer set defined by a maximal subset of linearly independent constraints of the linear relaxation and the original integrality restrictions (the remaining linear constraints are dropped). These relaxations generalize the

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corner relaxation introduced by Gomory [16], as they also consider infeasible bases of the linear relaxation. When a mixed-integer set is defined by linear equations and nonnegativity constraints on some variables, any split cut can be obtained as a *mixed-integer rounding* (MIR) inequality, as described by Nemhauser and Wolsey; see [17,2]. MIR inequalities are obtained by using nonnegativity constraints together with a single equation obtained as a linear combination of (a linearly independent subset of) the constraints of the linear relaxation. Therefore, depending on how the linear relaxation of the set is defined, it is possible to view split cuts as valid inequalities obtained from basic relaxations, or, as cuts obtained from 1-row relaxations.

Recently, split cuts have been generalized in different ways to obtain more effective cutting planes. One such generalization is to use two or more split disjunctions simultaneously to obtain valid inequalities. This gives rise to multi-branch split cuts, or *t-branch split cuts* when t split disjunctions are used. These cuts were first studied by Li and Richard [18] and recently Dash and Günlük extended some of their results [19]. Dash, Dey and Günlük [20,21] study 2-branch split cuts (and call them *cross cuts*) and *crooked cross cuts* (which are derived using three linearly dependent split disjunctions). Crooked cross cuts subsume cross cuts and are implied by 3-branch split cuts.

A different generalization of split cuts is obtained by considering multi-row relaxations of the mixed-integer set instead of one-row relaxations. This approach was introduced by Andersen, Louveaux, Weismantel and Wolsey [22] who study the so-called two-row continuous group relaxation and show that the convex hull of solutions of this relaxation is given by (two-dimensional) lattice-free cuts. This topic has received significant attention lately; see [23] for a recent survey. More generally, a k-row relaxation of a mixed-integer set is constructed by aggregating the equations defining the linear relaxation into k equations.

In this paper we compare cuts obtained from different generalizations of split cuts. In particular, we compare the strength of split, cross, crooked cross and general t-branch split cuts as well as cuts obtained from multi-row and basic relaxations, by comparing their elementary closures. As mentioned earlier, some results comparing the strength of these closures are already present in the literature; we next review some of these results and highlight our contributions in this paper. We say that a family of cuts *dominates* another if for every mixed-integer set, the elementary closure of the first family of cuts for the set is contained in the elementary closure of the second family of cuts for the same set. We say that the dominance is *strict* if there are examples where the elementary closure of the first family is strictly contained in the elementary closure of the second family. Henceforth, we refer to the elementary closure of a family $\mathcal{F}$ of cuts for a mixed-integer set as its $\mathcal{F}$-closure. Further, we define the *second $\mathcal{F}$-closure* as the elementary closure of the family $\mathcal{F}$ of cuts for the mixed-integer set defined by the constraints in the $\mathcal{F}$-closure of the original mixed-integer set along with its integrality restrictions. A cut in the family $\mathcal{F}$ that is valid for the second $\mathcal{F}$-closure is called a *rank-2 $\mathcal{F}$-cut*.

We next give an overview of the main results in the paper. The statements of the theorems are stated more formally in later sections.

1.1. Multi-branch split cuts

Recall that split cuts are the same as 1-branch split cuts and cross cuts are the same as 2-branch split cuts. Cook, Kannan and Schrijver [1, Example 2] presented a simple mixed-integer set (with two integer variables and one continuous variable) such that its convex hull is strictly contained in its second split closure (actually in its nth split closure for any finite n). The convex hull of this mixed-integer set equals its cross closure (by results in [22,20]). However, it is not known if in general cross cuts strictly dominate rank-2 split cuts. We answer this question and show that there is no such dominance relationship.

Theorem 1.1. *For every finite integer $t > 0$, there is a mixed-integer set whose second split closure is strictly contained in its t-branch split closure.*

Li and Richard [18] showed that for $t > 2$, t-branch split cuts strictly dominate 2-branch split cuts. Subsequently, it was shown in [19] that t-branch split cuts strictly dominate k-branch split cuts for all $t > k > 0$. In addition, it is known that 3-branch split cuts dominate crooked cross cuts which, in turn, dominate cross cuts [21,20]. However, these two dominance relationships are not known to be strict. In [21] the authors show that there is a crooked cross cut that is not implied by a *single* cross cut; however, this result does not rule out the possibility that the cross closure (which potentially contains infinitely many cuts) is *always equal* to the crooked cross closure. In this paper we establish that 3-branch split cuts strictly dominate crooked cross cuts which, in turn, strictly dominate 2-branch split cuts.

Theorem 1.2. *There is a mixed-integer set such that its crooked cross closure is strictly contained in its cross closure.*

Theorem 1.3. *There is a mixed-integer set such that its 3-branch closure is strictly contained in its crooked cross closure.*

Observation 1.4. *It was proved in [1] that the split closure of a mixed-integer set $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$ is given by*

$$\bigcap_{\pi \in \mathbb{Z}^m} \{(x, y) \in P : \pi x \in \mathbb{Z}\}.$$