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Advances on defective parameters in graphs[☆]

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HIGHLIGHTS

- k-sparse and k-dense sets generalize respectively independent sets and cliques.
- Defective Ramsey numbers avoiding k-sparse and k-dense sets are studied.
- Lower and upper bounds for unknown defective Ramsey Numbers are computed.
- The problem of partitioning graphs into k-sparse or k-dense sets is studied.
- Efficient graph generation methods are used to compute some defective parameters.

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ABSTRACT

We consider the generalization of Ramsey numbers to the defective framework using k-dense and k-sparse sets. We construct the first tableaux for defective Ramsey numbers with exact values whenever it is known, and lower and upper bounds otherwise. In light of defective Ramsey numbers, we consider the defective cocoloring problem which consists of partitioning the vertex set of a given graph into k-sparse and k-dense sets. By the help of efficient graph generation methods, we show that $c_0(4) = 12$, $c_1(3) = 12$ and $c_2(2) = 10$ where $c_k(m)$ is the maximum order n such that all n-graphs can be k-defectively cocolored using at most m colors. We also give the numbers of k-defective m-cocritical graphs of order n (until $n = 10$) for different levels of defectiveness and $m = 2, 3$ and 4.

1. Introduction

The classical Ramsey number $R(a, b)$ is defined as the minimum graph order n such that every graph of order at least n contains a clique of order a or an independent set of order b. One way to generalize Ramsey numbers is to introduce some defectiveness level k into the cliques and independent sets as defined in [1]. Given a graph G and an integer k, a set S of b vertices in G is called a *k-sparse b-set* if S induces a graph with degree at most k. Analogously, a set D of a vertices is a *k-dense a-set* if D induces a k-sparse a-set in the complement of G; in other words, in the graph induced by D, each vertex *misses* at most k other vertices in its neighborhood. Also, a set which is either k-dense or k-sparse is called *k-defective*. By abuse of language, we also say that a graph G is k-dense or k-sparse if the set of all vertices in G is k-dense or k-sparse, respectively. Then, the *defective Ramsey number $R_k(a, b)$*, is the minimum graph order n such that every graph of order at least n contains a k-dense a-set or a k-sparse b-set [1]. The defective Ramsey numbers can be seen as graph Ramsey numbers where graphs to

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be avoided are defined via the notion of defectiveness. See [2] for a comprehensive survey on such graph Ramsey numbers. The defective Ramsey numbers are also closely related to Ramsey numbers for graph sets with fixed number of edges [3]. In our case, instead of fixing a specific number of edges, the graphs to be avoided are defined by restricting the vertex degrees. In what follows, we will also use the notation n-graph, n-clique and n-independent set to denote respectively a graph, a clique and an independent set of order n.

Although defective coloring problems are extensively studied (see for instance [4]), there are only very few work on defective Ramsey numbers. In [5], only 1-defective Ramsey numbers are considered under the name of 1-dependent Ramsey numbers, and the values $R_1(4,5) = 9$, $R_1(4,6) = 11$, $R_1(4,7) = 16$, $R_1(4,8) = 17$ and $R_1(5,5) = 15$ are obtained. In [1], $R_2(5,5) = 7$ is shown. More recently, Chappell and Gimbel [6] derive some bounds on the defective Ramsey numbers, study their asymptotic behavior and obtain some exact values of defective Ramsey numbers. Among these values, only $R_2(5,6) = 8$ has a graph theoretical proof whilst the remaining are obtained by computer based generation of graphs. As it can also be observed in other studies about Ramsey numbers (see for instance [7,8]), the limits of mathematical proofs sees to be attained when it comes to the computation of new exact values of defective Ramsey numbers. In Section 2 of the present work, we give tableaux for defective Ramsey numbers where not only known exact values are shown (as in [6]) but also the best known lower and upper bounds are highlighted. These bounds are computed based on the results in [6] and then improved by the help of a graph generator. Although they do not offer any new exact value, they constitute, to the best of our knowledge, the first complete tableaux with best known bounds on defective Ramsey numbers.

As Ramsey numbers are closely related to the cocoloring problem, there is also a close relationship between defective Ramsey numbers and the so-called defective cocolorings. Given a graph G, an *m-cocoloring* of G is a partition of the vertex set of G into m subgraphs, such that each subgraph is a clique or an independent set. The cocoloring problem, defined in [9], is extensively studied in the literature not only in the past [10–13], but also quite recently [14–16]. A natural generalization of cocoloring to the defective framework is as follows: Given a graph G, a *k-defective m-cocoloring* of G consists in a partition of the vertex set of G into m subsets such that each one is a k-sparse or a k-dense set. The minimum number of sets in a k-defective cocoloring of G is the *k-defective cochromatic number* of G and denoted by $z_k(G)$ [1]. A graph G is called *k-defective m-cocritical* if $z_k(G) = m$ and the removal of any vertex from G results in a graph whose k-defective cochromatic number is $m - 1$. Note that for defectiveness level $k = 0$, we get the classical version of all the above parameters.

Now, let us define the parameter $c_k(m)$ as the maximum order n such that every n-graph has a k-defective m-cocoloring. The classical version of this parameter (when $k = 0$) was first studied in [10]. In the same paper, a formula (referred to as Straight's formula throughout this paper) relating Ramsey numbers to this parameter is developed and $c_0(3) = 8$ is shown whereas the determination of the exact values of $c_0(4) \in \{11, 12\}$ and $c_0(5) \in \{14, 15\}$ are left as open questions. As for $k > 0$, it is shown in [1] that $c_1(2) = 7$, $c_1(3) \geq 11$ and $c_2(2) \geq 9$. It seems that, exactly as for the Ramsey numbers, the limits of mathematical proofs are attained when it comes to the determination of new exact values for the parameter $c_k(m)$. In Section 3, we first generalize Straight's formula to the defective case, then we set $c_0(4) = 12$, $c_1(3) = 12$ and $c_2(2) = 10$ by computer assisted generation of graphs. Here, it is important to note that the number of graphs generated and checks effectuated grows exponentially with the order of the graph. For instance, in order to show $c_0(4) = 12$, we need to check all non-isomorphic graphs of order 12 (there are 165 091 172 592 of them) whether they are 4-cocolorable or not. If a 12-graph contains a 4-clique or a 4-independent set, then it is clearly 4-cocolorable since $c_0(3) = 8$. Another way a 12-graph is 4-cocolorable is that its vertex set can be partitioned into 4 cliques or independent sets, each one of size 3. Thus, for only one 12-graph, we have to check $\binom{12}{4} + \binom{12}{3}\binom{9}{3}\binom{6}{3} = 495 + 369600$ different subgraph combinations if we do not make any eliminations. Obviously, such computations start to be beyond the capacity of today's computers as the order of graphs to be generated grows. The crucial point in our algorithms is to narrow down the search space in a considerable way by the help of defective Ramsey numbers. As a byproduct, we also give the number of (non-isomorphic) k-defective m-cocritical graphs on up to 10 vertices for $m \in \{2, 3, 4\}$ and various defectiveness levels. Note that in [13], (0-defective) 3-cocritical graphs are characterized and, to the best of our knowledge, this is the only work on cocritical graphs in the literature. In Section 4, we conclude with some research directions.

All the algorithms in this paper are implemented in R programming language (except the improvement phase for the lower bounds on $R_k(a, b)$ which is implemented in Python) and executed on an Intel Core i7 machine with a 2.20-GHz clock speed and eight Gbyte of RAM memory.

2. Defective Ramsey numbers

The following bounds for defective Ramsey numbers are given in [6].

Proposition 2.1. *If* $a \geq 1$ *and* $b \geq k + 2$*, then* $R_k(a, b) \geq R_k(a - 1, b) + 1$.

Proposition 2.2. *If* $a \geq 2k + 1$ *and* $c, d \geq 1$*, then* $R_k(a, c + d - 1) \geq R_k(a, c) + R_k(a, d) + 1$.

Proposition 2.3. *If* $a, b \geq 2$*, then* $R_k(k + a, k + b) \geq k + a + b - 2$.

Proposition 2.4. *If* $a, b \geq 1$*, then* $R_k(a, b) \leq R_k(a - 1, b) + R_k(a, b - 1)$.