Discrete Optimization 6 (2009) 162–173

Contents lists available at ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

On the set covering polyhedron of circulant matrices

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ARTICLE INFO

Article history:
Received 6 March 2008
Received in revised form 31 October 2008
Accepted 5 November 2008
Available online 12 December 2008

Keywords:
Set covering polyhedron
Circulant matrix
Fractional extreme point
Non-Boolean facet

ABSTRACT

A well known family of minimally nonideal matrices is the family of the incidence matrices of chordless odd cycles. A natural generalization of these matrices is given by the family of circulant matrices. Ideal and minimally nonideal circulant matrices have been completely identified by Cornuéjols and Novick [G. Cornuéjols, B. Novick, Ideal 0 - 1 matrices, Journal of Combinatorial Theory B 60 (1994) 145–157]. In this work we classify circulant matrices and their blockers in terms of the inequalities involved in their set covering polyhedra. We exploit the results due to Cornuéjols and Novick in the above-cited reference for describing the set covering polyhedron of blockers of circulant matrices. Finally, we point out that the results found on circulant matrices and their blockers present a remarkable analogy with a similar analysis of webs and antiwebs due to Pêcher and Wagler [A. Pêcher, A. Wagler, A construction for non-rank facets of stable set polytopes of webs, European Journal of Combinatorics 27 (2006) 1172–1185; A. Pêcher, A. Wagler, Almost all webs are not rank-perfect, Mathematical Programming Series B 105 (2006) 311–328] and Wagler [A. Wagler, Relaxing perfectness: Which graphs are 'Almost' perfect?, in: M. Groetschel (Ed.), The Sharpest Cut, Impact of Manfred Padberg and his work, in: SIAM/MPS Series on Optimization, vol. 4, Philadelphia, 2004; A. Wagler, Antiwebs are rank-perfect, 4OR 2 (2004) 149–152].

1. Introduction

A graph is perfect if, in every vertex induced subgraph, the chromatic number equals the size of a largest clique. If M is the *clique-vertex* matrix of a perfect graph G, the polytope $QSTAB(G) = \{x \geq 0 : Mx \leq 1\}$ is integral, i.e. its extreme points are exactly the incidence vectors of the stable sets in G (see [4]). In this case the *stable set polytope* $STAB(G)$ of G, defined as the convex hull of the incidence vectors of all stable sets in G, coincides with $\{x \geq 0 : Mx \leq 1\}$. For the last forty years one of the most challenging open questions was to characterize the graphs that are not perfect but for which all proper vertex induced subgraphs are. These graphs are called *minimally imperfect* graphs. The Strong Perfect Graph Theorem, recently proved [3], states that odd holes and their complements (odd antiholes) are the only minimally imperfect graphs.

Graphs with circular symmetry of their maximum cliques and stable sets are called *webs*: a web W_n^k, with $k \geq 1$ and $n \geq 2(k+1)$, is a graph with vertices $1, \ldots, n$ where ij is an edge if i and j differ by at most $k (\text{mod } n)$ and $i \neq j$. Odd holes W^1_{2k+1} and odd antiholes W^{k-1}_{2k+1} are special kinds of webs. Webs are considered as generalizations of minimally imperfect graphs, and the status of imperfections of webs and antiwebs (complements of webs) has been widely studied, see [13,14,20,21] among others.

On the other hand, if M is a $0 - 1$ matrix and the polyhedron

$$Q(M) = \{x \geq 0 : Mx \geq 1\}, \tag{1}$$

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doi:10.1016/j.disopt.2008.11.001

is integral, the matrix M is called *ideal*. In this case $Q(M)$ coincides with the *set covering polyhedron* $Q^*(M)$ defined as the convex hull of the incidence vectors of all covers of M. Also, a matrix M is *minimally nonideal* if it is not ideal but for every $i \in \{1, \dots, n\}$, both $Q(M) \cap \{x : x_i = 0\}$ and $Q(M) \cap \{x : x_i = 1\}$ have no fractional extreme points.

The study of ideal matrices is not as advanced as that of perfect ones and it is apparently more difficult. Moreover, we lack a good understanding of the structure of minimally nonideal matrices. The starting point of the study of these matrices is Lehman's work [8] and [9]. In fact, in these works, Lehman called these matrices as *width–length* matrices, but Cornuéjols and Novick renamed them as ideal matrices "to stress the analogy with perfect matrices", [6]. In particular, in [8], three infinite classes of minimally nonideal matrices were presented, one of them is the well known family of odd holes and *circulant matrices* are their natural generalization.

The goal of this work is to show that many polyhedral aspects associated with the stable set polytopes of webs and antiwebs have their counterpart in the set covering polyhedra associated with circulant matrices and their blockers.

This paper is organized as follows: In the next section we state the notation, definitions and previous results we need in this work and we introduce the class of *rank-ideal* matrices. In Section 3 we introduce circulant matrices and study their status of nonidealness. In Section 4, with the help of results due to Cornuéjols and Novick in [6], we characterize all fractional extreme points of $Q(M)$ when M is a circulant matrix. This characterization completely determines the status of nonidealness of the blockers of circulant matrices. In Section 5, we prove that almost all circulant matrices are not rank-ideal. In Section 6, we enlarge the family of non-Boolean facets for the set covering polyhedra of circulant matrices found by Nobili and Sassano in [10]. Finally, in Section 7, we show the remarkable analogy between the classification of circulant matrices and their blockers done in this work and a similar study on webs and antiwebs due to Pêcher and Wagler in [13,14], and Wagler in [20,21].

2. Preliminary definitions and results

A *clutter* C is a pair $(V(C), E(C))$, where $V(C)$ is a finite set and $E(C)$ is a family of subsets of $V(C)$ none of which is included in another. The elements of $V(C)$ and $E(C)$ are the *vertices* and the *edges* of C, respectively. A clutter C is *trivial* if it has no edge or if $\emptyset$ is its unique edge. In the following, whenever the meaning is clear from the context, we assume $V(C) = V = \{1, \dots, n\}$, $E(C) = E$ and $|E| = m$.

Given $j \in V$, the clutter C/j is defined as follows: $V(C/j)$ is $V - \{j\}$ and $E(C/j)$ is the set of minimal elements of $\{S - \{j\} : S \in E\}$. We say that C/j is obtained by *contraction* of j. The clutter $C \setminus j$ is defined by $V(C \setminus j) = V - \{j\}$ and $E(C \setminus j) = \{S \in E : j \notin S\}$. We say that $C \setminus j$ was obtained by *deletion of* j. It is straightforward to check that if V_1 and V_2 are disjoint sets of vertices in V, contracting all vertices in V_1 and deleting all vertices in V_2 can be performed sequentially, and the resulting clutter does not depend on the order of the operations or vertices. Therefore, we can denote such a clutter by $C/V_1 \setminus V_2$ without any ambiguity. A *minor* of C is any clutter obtained from C by a sequence of deletions and contractions.

A *cover* of C is a set of vertices that intersects all edges of C. The *blocker* of C is the clutter $b(C)$ such that $V(b(C)) = V$ and $E(b(C))$ is the set of the minimal covers of C. It is known that for any clutter C, $b(b(C)) = C$, $b(C/i) = b(C) \setminus i$ and $b(C \setminus i) = b(C)/i$ for all $i \in V$ (see [5] for further details). The *covering number* of C, denoted by $\tau(C)$, is the minimum cardinality of a cover of C.

If C is a non-trivial clutter, $M(C)$ is the $0-1$ matrix whose rows are the characteristic vectors of the edges of C. Conversely, given a $0 - 1$ matrix M without dominating rows, i.e. a *clutter matrix*, there always exists a clutter C such that $M = M(C)$. In the following we will work with clutter matrices.

Given $M = M(C)$ we will denote by $M/V_1 \setminus V_2$ the matrix $M(C/V_1 \setminus V_2)$ and we will say that $M/V_1 \setminus V_2$ is a minor of M. From a polyhedral point of view, given $i \in \{1, \dots, n\}$, contracting (deleting) column i corresponds to adding $x_i = 0$ ($x_i = 1$, respectively) to the constraints given in (1) (see [5]).

Also, a cover of M is the incidence vector of a cover of C and $\tau(M) = \tau(C)$. In addition, the blocker of M is $b(M) = M(b(C))$.

If a matrix M is ideal so are its blocker [8] and all its minors [17].

A matrix M is *minimally nonideal* (mni, for short) if it is not ideal but all its proper minors are. The blocker of a mni matrix is also mni (see [9]).

Although a complete list of mni matrices is not known, all of them present interesting regularities (see [8,9]), except for the matrices associated to the clutter family J_n, $n \geq 3$, defined by $V(J_n) = \{0, 1, 2, \dots, n\}$ and $E(J_n) = \{\{1, 2, \dots, n\}, \{0, 1\}, \dots, \{0, n\}\}$. We will refer to mni matrices different from J_n as *regular* mni matrices. It is easy to check that $b(J_n) = J_n$, hence the blocker of a regular mni matrix is also regular mni.

In [12], Padberg proved that if M is a regular minimally nonideal matrix, then

$$Q^*(M) = Q(M) \cap \left\{x \in \mathbb{R}^n_+ : \mathbf{1}x \geq \tau(M)\right\}. \tag{2}$$

Following Sassano [16], we will call the restriction $\mathbf{1}x \geq \tau(M)$ the *rank constraint* associated with M.

In order to give some insight into mni matrices, in a previous work [2] we call a matrix M *near-ideal* if $Q^*(M)$ satisfies (2), i.e. the set covering polyhedron of M can be completely described by the inequalities in $Q(M)$ and the *rank constraint* associated with M. The class of near-ideal matrices strictly contains regular mni matrices and there are several properties that near-ideal matrices share with mni matrices. In particular:

Lemma 2.1 ([2]). *If M is a near-ideal matrix then $M \setminus i$ is ideal for all $i \in \{1, \dots, n\}$.*