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Valid inequalities and branch-and-cut for the clique pricing problem

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ABSTRACT

Motivated by an application in highway pricing, we consider the problem that consists in setting profit-maximizing tolls on a clique subset of a multicommodity transportation network. We formulate the problem as a linear mixed integer program and propose strong valid inequalities, some of which define facets of the two-commodity polyhedron. The numerical efficiency of these inequalities is assessed by embedding them within a branch-and-cut framework.

1. Introduction

The paradigm of pricing, either for improving the performance of infrastructures, or for maximizing the revenue of a private firm, pervades the economics literature. In the present paper, we consider the problem faced by a highway manager that seeks to maximize the revenue raised from tolls set on a network, while anticipating that users will travel on paths that maximize their individual utilities. This situation is closely related to the problem known as 'product line pricing' (see [1–5]), which is challenging from both the theoretical and computational points of view. Some years ago, Labbé et al. [6] recognized that the network pricing problem fits the framework of bilevel programming, a branch of optimization concerned with the solution of nonconvex programs involving two noncooperative agents, and that is akin to a leader–follower, or Stackelberg, game. This approach led to studies that focused on the combinatorial nature of network pricing, either in its original formulation or variants thereof. Representative of this approach are the works of Bouhtou et al. [7], van Hoesel et al. [8], Grigoriev et al. [9], Heilporn et al. [10,11], Kohli and Krishnamurti [12] and Roch et al. [13].

In the present paper, we consider a variant of the problem where all roads controlled by an authority are connected and form a path, as occurs in a toll highway. Assuming that tolls are levied with respect to all possible combinations of entry and exit points on the highway, one may focus on networks where a virtual arc is created for each entry–exit combination, and thus form an 'inner' clique. Shortest paths that do not go through the highway are represented by arcs linking the various origins and destinations, and form an 'outer' clique (see Fig. 1). The aim of this paper is to provide a better understanding of the Clique Pricing Problem and to develop algorithmic tools that can be transposed to situations arising in the field of revenue management (see [14]). More precisely, we are interested in the polyhedral structure of a specific Network Pricing

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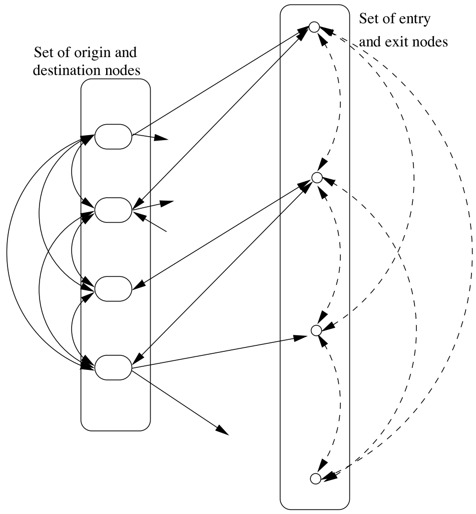

Fig. 1. Topology of the Clique Pricing Problem, where toll arcs are dashed and toll free arcs are solid. Nodes of the 'inner' clique (right) are the entry–exit nodes on the highway, while nodes of the 'outer' clique (left) represent various origins and destinations.

Problem. Preliminary results in that direction were obtained by Heilporn et al. [11], who provided a theoretical study of the single commodity Clique Pricing Problem.

The structure of the paper is as follows: Section 2 introduces the Clique Pricing Problem, together with its formulation; Section 3 deals with strong valid inequalities derived from the underlying network structure of the model; Section 4 provides proofs that the inequalities, as well as several constraints of the initial model, define facets of the two-commodity problem; finally, numerical results (Section 5) show that several of the valid inequalities are efficient, in the sense that their integration within a branch-and-cut scheme decreases the integrality gap, the CPU time, and the number of nodes explored in the resulting implicit enumeration process.

2. Mathematical formulation of the Clique Pricing Problem

Let us consider a linear highway composed of n entry–exit nodes, over which may transit m commodities, each of them associated with an origin–destination pair $k \in \mathcal{K}$ and a demand η^k. To each entry–exit pair correspond an arc $a \in \mathcal{A}$ and a commodity-specific cost $c_a^k + t_a$, where t_a denotes the toll. Commodities can either transit through the toll network, at cost $c_a^k + t_a$, or use alternative direct paths at cost u^k. Hence commodities who travel through the highway choose a single toll arc, i.e., they are not allowed to leave the highway and re-enter it later on. Assuming that all combinations of origin–destination and entry–exit nodes are present, the topology of the network is that of two cliques linked by 'access nodes' (see Fig. 1). The 'outer' clique is related to demand, while the 'inner' clique is a representation of the linear highway network.

Following Dewez [15], the Clique Pricing Problem can be formulated as the bilevel program:

$$\mathcal{CPP}: \quad \max_{t,x} \sum_{k\in\mathcal{K}} \sum_{a\in\mathcal{A}} \eta^k t_a x_a^k \tag{1}$$

subject to:

$$t_a \geq 0 \quad \forall a \in \mathcal{A} \tag{2}$$

$$(x, y) \in \arg\min_{\bar{x},\bar{y}} \sum_{k\in\mathcal{K}} \left(\sum_{a\in\mathcal{A}} (c_a^k + t_a)\bar{x}_a^k + u^k \bar{y}^k \right) \tag{3}$$

subject to:

$$\sum_{a\in\mathcal{A}} \bar{x}_a^k + \bar{y}^k = 1 \quad \forall k \in \mathcal{K} \tag{4}$$

$$\bar{x}_a^k \in \{0, 1\} \quad \forall k \in \mathcal{K},\ \forall a \in \mathcal{A} \tag{5}$$

$$\bar{y}^k \geq 0 \quad \forall k \in \mathcal{K}. \tag{6}$$

At the upper level, the authority seeks to maximize the profits earned by imposing tolls t_a on the inner clique's arcs. At the lower level, commodities are assigned to shortest paths with respect to the sum of fixed costs and tolls. The flow constraints (4) ensure that each commodity $k \in \mathcal{K}$ is assigned either to a toll path a of the inner clique ($x_a^k = 1$), or to a toll free path of the outer clique ($y^k = 1$). Note that, since we assume that any commodity chooses the shortest path from