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Circuit and bond polytopes on series–parallel graphs[☆]

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ABSTRACT

In this paper, we describe the circuit polytope on series–parallel graphs. We first show the existence of a compact extended formulation. Though not being explicit, its construction process helps us to inductively provide the description in the original space. As a consequence, using the link between bonds and circuits in planar graphs, we also describe the bond polytope on series–parallel graphs.

In an undirected graph, a *circuit* is a subset of edges inducing a connected subgraph in which every vertex has degree two. In the literature, a circuit is sometimes called *simple cycle*. Given a graph and costs on its edges, the *circuit problem* consists in finding a circuit of maximum cost. This problem is already NP-hard in planar graphs [1], yet some polynomial cases are known, for instance when the costs are non-positive.

Although characterizing a polytope corresponding to an NP-hard problem is unlikely, a partial description may be sufficient to develop an efficient polyhedral approach. Concerning the *circuit polytope*, which is the convex hull of the (edge-)incidence vectors of the circuits of the graph, facets have been exhibited by Bauer [2] and Coullard and Pulleyblank [3], and the cone has been characterized by Seymour [4]. Several variants of cardinality constrained versions have been studied, such as [5–8].

For a better understanding of the circuit polytope on planar graphs, a natural first step is to study it in smaller classes of graphs. For instance, in [3], the authors provide a complete description in Halin graphs.

Another interesting subclass of planar graphs are the series–parallel graphs. Due to their nice decomposition properties, many problems NP-hard in general are polynomial for these graphs, in which case it is quite

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standard to (try to) characterize the corresponding polytopes. Results of this flavor were obtained for various combinatorial optimization problems, such as the stable set problem [9], graph partitioning problem [10], 2-connected and 2-edge-connected subgraph problems [11,12], k-edge-connected problems [13], Steiner-TSP problem [14].

Since a linear time combinatorial algorithm solves the circuit problem in series–parallel graphs, an obvious question arising is the description of the corresponding polytope. Surprisingly, it does not appear in the literature, and we fill in this gap with Theorem 11.

The main ingredient for the proof of our main theorem is the existence of a compact extended formulation for the circuit polytope on series–parallel graphs. An *extended formulation* of a given polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polyhedron $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : Bx + Cy \leq d\}$ whose projection onto the x variables $proj_x(Q) = \{x \in \mathbb{R}^n :$ there exists $y \in \mathbb{R}^m$ such that $(x, y) \in Q\}$ is P. The *size* of a polyhedron is the number of inequalities needed to describe it. An extended formulation is called *compact* when its size is polynomial. We refer to [15] for further insights on this topic.

The past few years, extended formulations proved to be a powerful tool for polyhedral optimization, and thus received a growing interest in the community. Indeed, describing a polytope directly in its original space is often pretty challenging, and by looking for an extended formulation one has more tools at disposal. As an example, for most combinatorial optimization polytopes in series–parallel graphs, Martin et al. [16] proposed a general technique to derive extended formulations from dynamic programming algorithms, but the corresponding descriptions in the original space remain unknown.

Recently, it has been shown that the perfect matching polytope admits no compact extended formulation [17]. It means, even if an optimization problem is polynomial, there may not exist such a formulation. Here, though we are not able to explicitly construct a compact extended formulation for the circuit polytope on series–parallel graphs, we show that there exists one, see Section 2.1.1. The construction process of this extended formulation relies on a straightforward inductive description of the circuits of series–parallel graphs, combined with a theorem of Balas [18,19]. It allows us to prove by induction that the circuit polytope on series–parallel graphs is completely described by three families of inequalities. We provide examples where exponentially many of these inequalities define facets, see Corollary 19. Thus, the circuit polytope on series–parallel graphs is another example of polytope having exponentially many facet-defining inequalities that admits a compact extended formulation.

A graph is series–parallel if and only if, given any planar drawing of the graph, its dual is series–parallel. The dual of a circuit is a *bond*, that is a cut containing no other nonempty cut. These bonds play an important role e.g. in multiflow problems [20]. By planar duality and the description of the circuit polytope on series–parallel graphs, we get the description of the bond polytope on series–parallel graphs, see Theorem 13.

The paper is organized as follows. In Section 1, we fix graph related notation and definitions, and review some known and new auxiliary results about circuits in series–parallel graphs. Section 2 deals with the circuit polytope on series–parallel graphs. First, we get a polyhedral description of the latter for non trivial 2-connected series–parallel graphs, by providing the existence of a compact extended formulation, and then inductively projecting it. By applying standard techniques, the polyhedral description for general series–parallel graphs follows, which has exponential size in general. In Section 3, using the planar duality, we describe the bond polytope on series–parallel graphs, and then we study facet-defining inequalities, which have counterparts for the circuit polytope as well.

1. Circuits in series–parallel graphs

Throughout, $G = (V, E)$ will denote a connected undirected graph with $n = |V|$ vertices and $m = |E|$ edges. The graph *induced* by a subset W of V is the graph $G[W]$ obtained by removing the vertices of $V \setminus W$, and $\delta_G(W)$ is the set of edges having exactly one extremity in W. Given disjoint $U, W \subset V$, $\delta_G(U, W)$ is