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The median game[☆]

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ABSTRACT

We introduce a game which is played by two players on a connected graph G. The players I and II alternatively choose vertices of the graph until all vertices are taken. The set of vertices chosen by player I is denoted by Π_I, and by II is denoted by Π_{II}. Let $d(x,\pi) = \sum_{y\in\pi} d(x,y)$ and let $M(\pi) = \min\{d(x,\pi) \mid x \in \pi\}$ be the *median value* of a profile $\pi \subseteq V(G)$. The objective of player I is to maximize $M(\pi_{II}) - M(\pi_I)$ and the objective of player II is to minimize $M(\pi_{II}) - M(\pi_I)$. The winner of the game is the player with the smaller median value of her profile. We give a necessary condition for a tree so that player I (who begins the game) has a winning strategy for the game. We prove also that for hypercubes and some other symmetric graphs the player II has a strategy to draw the game. Complete bipartite graphs are considered as well.

1. Introduction

Many problems in combinatorics have their origins in games. For example the famous problem of Hamiltonicity of a graph started as a game on the octahedron. The tower of Hanoi problem also started as a game where the player should arrange the discs in a prescribed order (see [1] for a comprehensive historical and mathematical insight). Also the motivation for the first publication in graph theory by Euler may be

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considered as a game, where the players objective is to find a way over all Königsberg bridges and return to the origin.

All games mentioned above are played by one person, however the standard notion of a game in game theory involves two players opposing each other in order to meet the objective of the game. One such game is the coloring game introduced by Gardner in [2], where the main problem is to determine the game chromatic number of a graph. This question was asked for many different classes of graphs, and in particular it was proved in [3] that the game chromatic number of a planar graph is at most 17.

Similar are the marking game and the greedy game. Both games were introduced by Zhu (see [4,5]), the marking game in context of studying the game chromatic number and the greedy game as an intermediate between the marking and the coloring game. There are also other well studied games on graphs, such as the cop and robber game (see [6–8]) and the domination game (see [9,10]).

The problem of finding an optimal location for placing a facility is a classical problem in optimization. The theoretical and algorithmical aspects of this problem are extensively studied in the literature, see [11–13]. The problem of locating the distribution center is conventionally modeled as the median location problem and the problem of locating median sets for profiles on graphs was studied by many authors with different assumptions (see, for example, [14–17]). In this paper we introduce a new concept where the main objective is to choose several locations so that the sum of distances to a fixed location (where, for example, a distribution center may be located) is as small as possible (and smaller than the sum of distances of the competition).

Similar model was introduced in [18], where the objective is to minimize the sum of distances between all pairs of vertices. This game is called the Wiener game in [18].

Let $G = (V, E)$ be a graph and $\pi \subseteq V$ be a set, which we call a *profile*. For vertices $x, y \in V(G)$ the *distance* between x and y, denoted as $d(x, y)$, is the number of edges on a shortest x, y-path in G. The *remoteness* (total distance) of a vertex x with respect to π is defined as

$$R(x) = d(x, \pi) = \sum_{y \in \pi} d(x, y).$$

The *median value* of π, denoted as $M(\pi)$, is the minimum remoteness of a vertex in π with respect to π. If $\pi = V(G)$, then $M(V(G))$ is called a *median value* of G, and each vertex for which $M(V(G))$ is achieved is called a *median vertex*.

The *median game* is played by two players I and II on a graph G. Player I starts and takes a vertex of G, then player II takes another vertex. The game continues this way until all vertices of G are taken. Let π_I and π_{II} denote the set of vertices that belong to players I and II at the end of the game, respectively. The objective of player I is to maximize the value $M(\pi_{II}) - M(\pi_I)$ and conversely the goal of player II is to minimize $M(\pi_{II}) - M(\pi_I)$. The winner of the game is the player with the smaller median value of his/her profile. That is, if $M(\pi_{II}) - M(\pi_I) > 0$, then player I wins, and if $M(\pi_{II}) - M(\pi_I) < 0$, then player II wins. If $M(\pi_{II}) - M(\pi_I) = 0$, then it is a draw.

If player I has a strategy such that $M(\pi_{II}) - M(\pi_I) \geq \mu$ for any strategy (choice of vertices) of player II, and conversely, if player II has a strategy such that $M(\pi_{II}) - M(\pi_I) \leq \mu$ for any strategy (choice of vertices) of player I, then we call μ the *median game number* of G, and denote it by $\mu(G)$. Hence $\mu(G) > 0$ means that player I has a winning strategy, $\mu(G) < 0$ means player II has a winning strategy, and $\mu(G) = 0$ means that both players have drawing strategies.

In game theory a game in which the players are concerned about maximizing his own profit (which is $M(\pi_X)$, $X \in \{I, II\}$ in this case) along with minimizing his opponent's profit uses a *mixed strategy*. Clearly median game is such a game as the objective of player I is to maximize $M(\pi_{II}) - M(\pi_I)$ and the objective of player II is vice versa to minimize $M(\pi_{II}) - M(\pi_I)$. The famous von Neumann's Min Max Theorem then imply that $\mu(G)$ exists for any finite graph G and hence $\mu(G)$ is well defined.

The order of a graph seems to play an important role for this game. Namely, if G has an even number of vertices, then both players obtain the same number of vertices, but if G is odd, then player I gets one