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Multi-commodity variable upper bound flow models[☆]

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We perform a polyhedral study of a multi-commodity generalization of variable upper bound flow models. In particular, we establish some relations between facets of single- and multi-commodity models. We then introduce a new family of inequalities, which generalizes traditional flow cover inequalities to the multi-commodity context. We present encouraging numerical results.

1. Introduction

Variable upper bound flow models are well-studied in the mixed-integer programming literature. The practical value of these models stems from the fact that they often occur as substructures of MILPs. In particular, they can be viewed as single-node instances of the fixed-charge capacitated network design problem, which has applications in telecommunications and transportation [1].

For the case where a single commodity is considered, Padberg, van Roy, and Wolsey [2] study the polyhedral structure of variable upper bound flow models and identify the family of flow cover inequalities. Wolsey [3] shows that these inequalities can be derived using properties of submodular functions. Gu, Nemhauser, and Savelsbergh [4] describe a procedure for lifting flow cover inequalities. Various authors have studied generalizations of the classic variable upper bound flow model, which in turn, yield extensions of flow cover inequalities. For instance, Klabjan and Nemhauser [5] study variable upper bound flow models with general integer variable upper bounds, while Shebalov and Klabjan [6] consider models in which the variable upper bound constraints include constant terms. Atamtürk, Nemhauser and Savelsbergh [7] examine the case with additive variable upper bound constraints.

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Various families of valid inequalities have been identified for variants of the multi-commodity capacitated network design problem (MCND). Bienstock et al. [8] apply *partition inequalities* and *flow-cutset inequalities* in a branch-and-cut framework to solve the minimum cost capacity installation problem. Günlük [9] uses mixing to obtain valid inequalities for the capacity expansion problem. Atamtürk [10] studies single- and multi-commodity cutset polyhedra arising from directed network design problems, while Raack et al. [11] consider cutset polyhedra for directed, undirected, and bi-directed arcs. These families of inequalities apply to network design problems with general integer design variables and therefore are valid for the 0–1 case. The model we study considers binary design variables specifically. We are therefore able to obtain families of facet-defining inequalities for the 0–1 *fixed-charge* problem described in [12].

More precisely, we perform a polyhedral study of a multi-commodity generalization of variable upper bound flow models. In Section 2, we present the model formulation, define notation used throughout the paper, and derive basic polyhedral results. In Section 3, we explore the use of commodity aggregation as a way to obtain strong valid inequalities for the model. In Section 4, we define hierarchical flow cover inequalities, which generalize flow cover inequalities. We provide a set of sufficient conditions under which these valid inequalities are facet-defining for the model. In Section 5, we lift hierarchical flow cover inequalities to obtain valid inequalities for a model with in- and outflows, and give conditions under which these lifted inequalities are facet-defining. We present computational results in Section 6, and give concluding remarks in Section 7.

2. Single node model without inflow

Consider first a network node with outgoing arcs $N := \{1, \ldots, |N|\}$ and a set of distinct commodities $K := \{1, \ldots, |K|\}$. For $k \in K$, the node has an exogenous supply in the amount of b^k, where $b^k \in \mathbb{R}_+$. Further, each arc $j \in N$ has capacity $m_j \in \mathbb{R}_+$. Commodities can be routed on any of the arcs leaving the node, as long as they are not overused, and the consolidated flow on each arc respects its capacity. For $j \in N$ and $k \in K$, we let the binary variable x_j correspond to the decision of whether or not to open arc j, and we let the continuous variable y_j^k represent the flow of commodity k on arc j. We will show in Section 5 that strong inequalities for this set directly yield strong inequalities for the model with additional incoming arcs.

The aforementioned model can be expressed mathematically as follows:

$$\sum_{j \in N} y_j^k \le b^k, \quad \forall k \in K, \tag{1a}$$

$$\sum_{k \in K} y_j^k \le m_j x_j, \quad \forall j \in N, \tag{1b}$$

$$x_j \le 1, \quad \forall j \in N, \tag{1c}$$

$$x_j \ge 0, \quad \forall j \in N, \tag{1d}$$

$$y_j^k \ge 0, \quad \forall j \in N, k \in K. \tag{1e}$$

Constraints (1a) guarantee that the flow of each commodity k does not exceed its supply. Constraints (1b) ensure that the capacity of each arc j is respected. Constraints (1c) and (1d) are upper and lower bounds constraints on the x_j variables, respectively. Constraints (1e) enforce the nonnegativity of the flow variables y_j^k.

We define the *multi-commodity variable upper bound flow model* (MVF) as the set

$$P := \left\{(x, y) \in \mathbb{Z}^N \times \mathbb{R}^{N \times K} \mid \text{(1a)–(1e)}\right\}.$$

We note that MVF reduces to a traditional variable upper bound flow model in the case that $|K| = 1$.