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Cardinality constrained combinatorial optimization: Complexity and polyhedra

Rüdiger Stephan

Center for Operations Research and Econometrics, Université catholique de Louvain, Voie de Roman Pays 34, B-1348 Louvain-la-Neuve, Belgium

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ABSTRACT

Given a combinatorial optimization problem and a subset N of nonnegative integer numbers, we obtain a cardinality constrained version of this problem by permitting only those feasible solutions whose cardinalities are elements of N. In this paper we briefly touch on questions that address common grounds and differences of the complexity of a combinatorial optimization problem and its cardinality constrained version. Afterwards we focus on the polyhedral aspects of the cardinality constrained combinatorial optimization problems. Maurras (1977) [5] introduced a class of inequalities, called *forbidden cardinality inequalities* in this paper, that can be added to a given integer programming formulation for a combinatorial optimization problem to obtain one for the cardinality restricted versions of this problem. Since the forbidden cardinality inequalities in their original form are mostly not facet defining for the associated polyhedron, we discuss some possibilities to strengthen them, based on the experiments made in Kaibel and Stephan (2007) and Maurras and Stephan (2009) [2,3].

1. Introduction

Given a combinatorial optimization problem and a finite subset N of the nonnegative integer numbers $\mathbb{Z}_+$, we obtain a cardinality constrained version of this problem by permitting only those feasible solutions whose cardinalities are elements of N.

Well-known examples of cardinality constrained combinatorial optimization problems are the traveling salesman problem and the minimum odd cycle problem. Both problems are for themselves combinatorial optimization problems, but in the line of sight of the minimum cycle problem, they are cardinality restricted version of the latter problem.

More formally, let E be a finite set, $\mathcal{I}$ a subset of the power set 2^E of E, and $w : E \to \mathbb{R},\ e \mapsto w(e)$ a weight function. For any $F \subseteq E$ and any $y \in \mathbb{R}^E$, we set $y(F) := \sum_{e\in F} y_e$. The mathematical program

$$\max\{w(F) : F \in \mathcal{I}\}$$

is called a *combinatorial optimization problem* (COP). We also refer to it as the triple $\Pi = (E, \mathcal{I}, w)$. The elements of $\mathcal{I}$ are called *feasible solutions*. By permitting only those feasible solutions whose cardinalities belong to a given finite set $N \subset \mathbb{Z}_+$, we obtain a cardinality constrained version $\Pi_N = (E, \mathcal{I}, w, N)$ of Π. The resulting problem is also called a *cardinality constrained combinatorial optimization problem* (CCCOP). Here, the *cardinality* of any finite set M, denoted by $|M|$, is the number of its elements. The cardinality constrained version Π_N of Π can be expressed as the mathematical program

$$\max\{w(F) : F \in \mathcal{I},\ |F| \in N\}.$$

We note that Π_N is, considered for itself, again a COP.

E-mail address: stephan@math.tu-berlin.de.

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Throughout this paper, N will be represented by a so-called *cardinality sequence*, which is a sequence $c = (c_1, c_2, \ldots, c_m)$ of integers such that $N = \{c_1, \ldots, c_m\}$ and $0 \leq c_1 < c_2 < \cdots c_m \leq |E|$. Moreover, Π_N will be identified with Π_c. The set of feasible solutions with respect to Π_c will also be denoted by $\mathcal{I}_c$, that is, $\mathcal{I}_c := \{I \in \mathcal{I} : |I| = c_p \text{ for some } p\}$. If $c = (k)$ for some $k \in \mathbb{Z}_+$, we speak of a k-COP and write Π_k instead of $\Pi_{(k)}$ provided that it is clear from the context that c refers to a sequence and k to an integer. An overview on k-COPs is given by Bruglieri et al. [1].

This paper focuses on polyhedral aspects of cardinality constrained combinatorial optimization problems, but also briefly addresses complexity issues.

Many combinatorial optimization problems are polyhedrally well studied. Given a COP $\Pi = (E, \mathcal{I}, w)$, the polyhedral investigation usually refers to the associated polytope $P_{\mathcal{I}}(E)$ defined as the convex hull of the incidence vectors χ^I of the feasible solutions $I \in \mathcal{I}$. In this paper, we study the polytope

$$P^c_{\mathcal{I}}(E) := \text{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I}_c\},$$

that is, the convex hull of the incidence vectors of feasible solutions with respect to Π_c. Since $\mathcal{I}_c \subseteq \mathcal{I}$, it follows that $P^c_{\mathcal{I}}(E) \subseteq P_{\mathcal{I}}(E)$. Thus, any valid inequality for $P_{\mathcal{I}}(E)$ is also valid for $P^c_{\mathcal{I}}(E)$. It stands to reason that many facet defining inequalities for $P_{\mathcal{I}}(E)$ are also facet defining or at least strong inequalities for $P^c_{\mathcal{I}}(E)$ (see, for instance, [2,3]). In this paper, however, we are more interested in strong valid inequalities that cut off solutions that are feasible for Π but forbidden for Π_c.

To the best of our knowledge, such inequalities, we are interested in, have been first introduced by Jeroslow [4] and Maurras [5] in the 1970's (see also [6]). In [4], it has been shown that (the convex hull of) all vertices of the unit hypercube $H \subseteq \mathbb{R}^n$ of even size (that is, with an even number of ones) are determined by the inequalities

$$\begin{aligned} & 0 \leq x_i \leq 1 \quad \text{for all } i \in [n], \\ & \sum_{i \in S} x_i - \sum_{i \in [n] \setminus S} x_i \leq |S| - 1 \quad \text{for all } S \subseteq [n],\ |S| \text{ odd}, \end{aligned}$$

where $[n] := \{1, 2, \ldots, n\}$. A generalization of this result can be found in [6,5]. Given a cardinality sequence $c = (c_1, c_2, \ldots, c_m)$, a complete linear description of the polytope H^c defined as the convex hull of all vertices of H of size c_p for some $p \in \{1, 2, \ldots, m\}$ is as follows:

$$0 \leq x_i \leq 1 \quad \text{for all } i \in [n], \tag{1}$$

$$c_1 \leq \sum_{i \in [n]} x_i \leq c_m, \tag{2}$$

$$\begin{aligned} & (c_{p+1} - |S|) \sum_{i \in S} x_i - (|S| - c_p) \sum_{i \in [n] \setminus S} x_i \leq c_p(c_{p+1} - |S|) \\ & \quad \text{for all } S \subseteq [n] \text{ with } c_p < |S| < c_{p+1},\ p = 1, 2, \ldots, m-1. \end{aligned} \tag{3}$$

We shall call inequalities (1) *trivial inequalities*, inequalities (2) *cardinality bounds*, and inequalities (3) *forbidden cardinality inequalities*. We note that the result for H^c has been rediscovered by Grötschel [7] in form of a linear description of *cardinality homogeneous set systems*, see Section 3.

Identifying $\mathbb{R}^n$ with $\mathbb{R}^E$, the linear description of H^c can be Incorporated in any integer programming formulation for a COP $\Pi = (E, \mathcal{I}, w)$ to obtain one for its cardinality constrained version. However, as it turns out, the resulting formulations can be become very weak. Nevertheless, recently it have been proposed strong integer programming formulations for cardinality constrained path and cycle problems defined on directed or undirected graphs (see [2]) and a linear program for the cardinality constrained version of the maximum independent set problem defined on matroids (see [3]). These formulations have been obtained by modifying inequalities (3).

Motivated by the results in [2,3], the general goal of this paper is to identify those features of cardinality constrained combinatorial optimization problems that, expressed in form of linear inequalities, result into strong integer programming formulations. Since this question is quite difficult to handle in general, we mainly analyze cardinality constrained independence systems and in particular matroids, and show how the resulting polyhedral insights can be transferred to other problems.

The paper is organized as follows. Before turning to polyhedral aspects of CCCOPs in more detail, we address, in Section 2, some complexity issues concerning the relation between the complexity of a COP and its cardinality constrained version. In Section 3, we first give two examples indicating that inequalities (3) might be quite weak, that is, that they define low-dimensional faces of the polyhedron considered. Next, we give three recommendations to strengthen them. As a result of one of these recommendations, we derive in Section 4 a class of facet defining inequalities for a cardinality constrained version of the cut polytope defined on a complete graph. In Section 5, we briefly touch the problem to derive a complete linear description of the polyhedra associated with Π_c provided we know a complete linear description of the polyhedron associated with Π.

2. Complexity issues

In this section, we briefly touch the question under which conditions a COP Π and its cardinality constrained version Π_c belong to the same complexity class. The aim of this discussion is not to give a concluding answer to this question, but to mark the challenges to answer this question if we do not study a specific COP.