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New upper and lower bounds for online scheduling with machine cost

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ABSTRACT

This paper considers the online scheduling problem with machine cost. We are given a sequence of independent jobs with positive sizes. Jobs come one by one and it is required to schedule jobs irrevocably to a machine as soon as they are given, without any knowledge about jobs that follow later on. No machines are initially provided. When a job is revealed, the algorithm has the option to purchase new machines. The objective is to minimize the sum of the makespan and cost of purchased machines. We prove that $\sqrt{2}$ is a lower bound of the problem, which significantly improves the previous one of $4/3$. We also present a new algorithm with competitive ratio $(2+\sqrt{7})/3 \approx 1.5486$, which improves the current best algorithm with competitive ratio $(2\sqrt{6}+3)/5 \approx 1.5798$. Moreover, we prove that applying only the lower bounds on the optimum objective value introduced before, no algorithm can be proven to have a competitive ratio less than $3/2$.

1. Introduction

This paper considers an online scheduling problem with machine cost, which was first studied in [1]. We are given a sequence of independent jobs $J_1, J_2, \ldots, J_n$ with positive sizes. Jobs come one by one and it is required to schedule jobs irrevocably to a machine as soon as they are given, without any knowledge about jobs that follow later on. Jobs are available at times zero, and no preemption is allowed. Unlike classical online parallel machine scheduling [2–5], no machines are initially provided. When a job is revealed, the algorithm has the option to purchase new machines. The cost of purchasing a machine is a fixed constant. The objective is to minimize the sum of the makespan and cost of purchased machines.

The quality of an online algorithm H is measured by its *competitive ratio*. For any sequence I of jobs, let $C^H(I)$ denote the corresponding objective value of a schedule produced by H, and $C^*(I)$ denote the optimal objective value. Then the competitive ratio of H is defined as the smallest number t such that $C^H(I) \leq tC^*(I)$ for all sequences. An algorithm with a competitive ratio at most t is called a *t-competitive algorithm*. An online scheduling problem has a *lower bound* ρ if no online algorithm has a competitive ratio smaller than ρ.

In [1], Imreh and Noga proved that $4/3$ is a lower bound for the problem. They also designed an online algorithm A_ρ with competitive ratio $(1+\sqrt{5})/2 \approx 1.618$. An improved algorithm with competitive ratio $(2\sqrt{6}+3)/5 \approx 1.5798$ is presented by Dósa and He [6]. Some variants of the problem have been studied in [7–9]. In [10], Imreh considered a more general model, where the cost of purchased machines is described by a non-decreasing function.

In this paper, we give improved lower and upper bounds for the scheduling problem with fixed machine cost 1. We prove that $\sqrt{2}-\varepsilon$ is a lower bound of the problem, where ε can be an arbitrarily small positive number. To the authors' knowledge, it is the first improvement on the lower bound over the past ten years. Then we present a more sophisticated algorithm,

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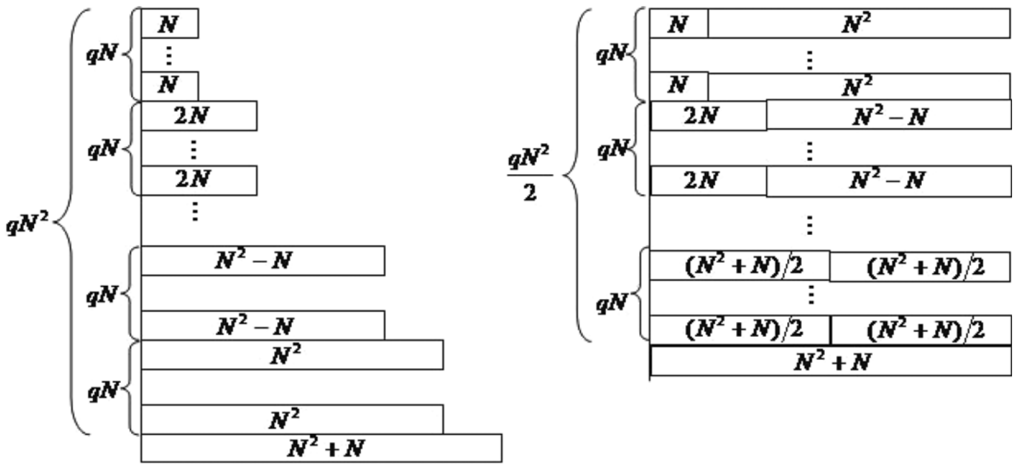

Fig. 1. Sequence used for proving Theorem 2.1 (N is an odd number). Left: Schedule produced by an online algorithm. Right: A better schedule.

which uses new machine purchasing strategies, and the competitive ratio is $(2+\sqrt{7})/3 \approx 1.5486$. It improves the best known upper bound, and the idea of the algorithm may be used to related problems as well.

The improvement may seem small. However, we prove that applying only the lower bounds on the optimum objective value introduced in [1] and used in the previous papers, no online algorithm can be proven to have a competitive ratio less than 3/2. This result strictly limits the performance of online algorithms that can be achieved using known technique. Note that the competitive ratio of our algorithm is only 0.05 larger. It seems that the performance of online algorithms can hardly be further improved unless new lower bounds on the optimum objective value can be found. Albers [4] gave similar results for classical online parallel machine scheduling. Applying only three lower bounds on the optimal makespan ever used, no online algorithm can be proven to have a competitive ratio less than 1.916, which is 0.04 smaller than the competitive ratio of the best known online algorithm [3].

The result of the paper is organized as follows. In Section 2 we present the new lower bounds. In Section 3 we give the description of the algorithm and some preliminary results. The competitive ratio of the algorithm is proved in Section 4.

2. Lower bounds

Theorem 2.1. *Any deterministic online algorithm has a competitive ratio of at least* $\sqrt{2}$.

Proof. We use the adversary method to get a new lower bound arbitrarily close to $\sqrt{2}$. Let ε be an arbitrarily small positive number. Let q be a rational number such that $\sqrt{2}-\frac{\varepsilon}{2} \le q < \sqrt{2}$, and N be a sufficiently big integer such that $qN > 4/\varepsilon$ and qN is an even number.

The sequence of jobs consists of at most qN^2+1 jobs (Fig. 1). All the jobs, except the last one, are divided into N successive batches. Each batch has qN jobs with the same size. The size of jobs in the ith batch is iN, $i = 1, \ldots, N$. We will prove that in order to be $(\sqrt{2}-\varepsilon)$-competitive, any algorithm A assigns each job to a new machine.

On the contrary, suppose the kth job in the jth batch is the first job that is assigned to an existing machine, $k = 1, \ldots, qN$, $j = 1, \ldots, N$. Then the sequence stops. The makespan of the current schedule is at least $N + jN$, the number of purchased machines is $(j-1)qN + k - 1$. Hence, $C^A \ge (j-1)\,qN + k - 1 + (j+1)\,N$. In a better schedule, each job in the lth batch shares one machine with one job in the $(j-l)$th batch, $l = 1, \ldots, \left\lfloor \frac{j-1}{2} \right\rfloor$. Two jobs in the $\frac{j}{2}$th batch share one machine when j is an even number. Each job in the jth batch occupies one machine. The load on each of the $(j-1)\frac{qN}{2} + k$ machines is jN. Hence, the objective value of above schedule is $(j-1)\frac{qN}{2} + k + jN$. It follows that

$$\begin{aligned}
\frac{C^A}{C^*} &\ge \frac{(j-1)\,qN + k - 1 + (j+1)\,N}{(j-1)\frac{qN}{2} + k + jN} \ge \frac{(j-1)\,qN + qN - 1 + (j+1)\,N}{(j-1)\frac{qN}{2} + qN + jN} \\
&= \frac{jq + (j+1)}{\frac{j+1}{2}q + j} - \frac{1}{\left(\frac{j+1}{2}q + j\right)N} > \frac{jq + (j+1)\frac{q^2}{2}}{j + (j+1)\frac{q}{2}} - \frac{1}{(q+1)N} \\
&= q - \frac{1}{(q+1)N} > q - \frac{\varepsilon}{2} \ge \sqrt{2} - \varepsilon,
\end{aligned}$$

where the second inequality is due to $k \le qN$, the third inequality is due to $q < \sqrt{2}$ and $j \ge 1$, and the second last inequality is due to $qN > \frac{4}{\varepsilon}$.