Discrete Optimization 7 (2010) 21–31

Contents lists available at ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

Computing Steiner points for gradient-constrained minimum networks

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ARTICLE INFO

Article history:
Received 13 November 2008
Received in revised form 8 October 2009
Accepted 21 October 2009
Available online 1 December 2009

Keywords:
Gradient constrained
Steiner tree
Underground mining

ABSTRACT

Let T_g be a *gradient-constrained minimum network*, that is, a minimum length network spanning a given point set in 3-dimensional space with edges that are constrained to have gradients no more than an upper bound m. Such networks occur in underground mines where the slope of the declines (tunnels) cannot be too steep due to haulage constraints. Typically the gradient is less than 1/7. By defining a new metric, the gradient metric, the problem of finding T_g can be approached as an unconstrained problem where embedded edges can be considered as straight but measured according to their gradients. All edges in T_g are *labelled* by their gradients, being $<m$, $=m$ or $>m$, in the gradient metric space. Computing Steiner points plays a central role in constructing locally minimum networks, where the topology is fixed. A degree-3 Steiner point is *labelled minimal* if the total length of the three adjacent edges is minimized for a given labelling. In this paper we derive the formulae for computing labelled minimal Steiner points. Then we develop an algorithm for computing locally minimal Steiner points based on information from the labellings of adjacent edges. We have tested this algorithm on uniformly distributed sets of points; our results help in finding gradient-constrained minimum networks.

1. Introduction

In this section we review some fundamental properties of Steiner minimum trees and gradient-constrained networks and give the terminology used in this paper. In addition, the underground mine design problem, which motivated this paper, will be briefly described.

1.1. The Steiner tree problem

Given a point set N in a metric space, the *Steiner tree problem* asks for a network T spanning N with minimum length. The solution T is a tree with vertex set $V \supseteq N$ that spans the points in N, which are called *terminals*. The points in $V \setminus N$ are added to shorten the network. They are called *Steiner points*. The graph structure of a network is referred to as its *topology* and denoted by t. Further discussion of the Steiner tree problem can be found in the book by Hwang et al. [1].

In the classical Steiner tree problem, N lies in the Euclidean d-space, ($d \geq 2$). In this case the following proposition holds.

Proposition 1.1. *Suppose T is a minimum length network in Euclidean space, then*

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1 NICTA is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program.

1572-5286/$ – see front matter
doi:10.1016/j.disopt.2009.10.001

(1) *T has a tree topology, called a* Steiner minimum tree,
(2) *all edges in T are straight line segments,*
(3) *any angle at a Steiner point is no more than* 120° *(*angle condition*),*
(4) *the degree of a Steiner point is at most three, that is, T has a Steiner topology and is called a* Steiner tree.

Where we wish to emphasize the dependence of T on N, or both N and t, the network will be denoted by $T(N)$ or $T(N, t)$, respectively. Similarly, if we want to further explicitly specify the dependence of the minimal tree on the positions of the Steiner points, the set $S = \{s_1, s_2, \ldots\}$, then we will write T as $T = T(N, t, S)$ where S is a third variable for the function T.

A Steiner tree $T = T(N, t)$ is *locally optimal* on N if it is the shortest among all those trees having the same topology t but where the positions of the Steiner points differ. Hence, finding a locally optimal Steiner tree is a continuous optimisation problem. However, a Steiner minimum tree T is a *globally optimal* tree, that is, optimal over all Steiner topologies t as well as over all possible positions for the Steiner points for each t. Note that the topology t is a discrete variable, hence the Steiner tree problem, as a global optimisation problem, is both a continuous as well as discrete optimisation problem. The Euclidean Steiner tree problem is NP-hard [1]; the number of topologies is exponential in the size of N.

Remark 1.1. The Euclidean Steiner tree problem is even harder than other well-known combinatorial network optimisation problems such as the travelling salesman problem due to this hybrid optimisation. It is more difficult in higher-dimensional Euclidean space than in the plane, as finding the locally minimum Steiner tree, even on a set of four points in Euclidean 3-space, is not algebraically solvable in general [2].

To date there are many variants on the classical Steiner tree problem with metrics related to their different applications, *e.g.* rectilinear Steiner trees and λ-trees in VLSI design [3,4], flow-dependent networks in communications [5] and Steiner minimum trees in molecular biology [6]. In recent years, a further variant has been the problem of finding Steiner minimum trees in 3-dimensional space in which the (absolute) gradients of all edges are no more than an upper bound m. The application is in underground mine access design [7–10] and is described further in Section 1.2. Such networks are called gradient-constrained minimum networks and this problem has been shown to be NP-hard [11]. In fact all of the aforementioned variants have been shown to be NP-hard.

1.2. Gradient metric and gradient-constrained networks

Let x_p, y_p, z_p denote the Cartesian coordinates of a point p in Euclidean space. By the gradient $g(pq)$ of an edge pq we mean the absolute value of the slope from $p = (x_p, y_p, z_p)$ to $q = (x_q, y_q, z_q)$, that is,

$$g(pq) \overset{\text{def}}{=} \frac{|z_q - z_p|}{\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}}.$$

Here, $|\cdot|$ denotes simply the absolute value. If $g(pq) \le m$, then pq is a straight line segment joining p and q and is referred to as *straight*. However, if $g(pq) > m$, then pq cannot be represented, or embedded in 3-dimensional Euclidean space, as a straight line segment without violating the gradient constraint. Instead it can be represented by a zig-zag line joining p and q with each segment having gradient m. Such edges are referred to as *bent*.

Suppose o is the origin and $p = (x_p, y_p, z_p),\ q = (x_q, y_q, z_q)$ are points in 3-space. The *gradient metric* can be defined in terms of the Euclidean and vertical metrics, denoted by $|\cdot|_{\text{e}}$ and $|\cdot|_{\text{v}}$ respectively:

$$|pq|_{\text{g}} = \begin{cases} |pq|_{\text{e}} = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2} & \text{if } g(pq) \le m, \\ |pq|_{\text{v}} = (\sqrt{1 + m^{-2}})|z_p - z_q| & \text{if } g(pq) \ge m. \end{cases} \tag{1}$$

It is easy to see that the unit ball for the gradient metric looks like a drum, that is, like a ball whose North and South poles are cut off by horizontal planes of equal distance from the ball's centre (Fig. 1). Therefore the gradient metric, and consequently the length function of a gradient-constrained network, is convex but not strictly convex. A convex set is strictly convex if the relative open line segment between any two points on the boundary of the convex set lies strictly in the interior of the convex set.

A *gradient-constrained minimum network* $T_g = T_g(N)$ is a minimum length network spanning a given (finite) point set N in 3-dimensional space with edges whose (absolute) gradients are all no more than an upper bound m. Such networks occur in underground mines where ore is accessed and hauled to the surface via a network of gently sloping declines (tunnels). The declines cannot be too steep (Fig. 2) as driving up steep inclines requires more fuel and there is more wear and tear on the trucks. The typical maximum gradient of the tunnels is about 1:7 ($\approx$0.14) [8,9]. These networks may be many kilometres long. The development costs of the declines and associated haulage costs over the life of a mine are a major part of the overall mine costs. Saving 10 m in length of a decline results in savings of about 100,000 US dollars over the life of the mine. Hence minimising the length of the network is key to the viability and profitability of an underground mine.