Discrete Optimization 9 (2012) 216–235

Contents lists available at SciVerse ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

Mixed n-step MIR inequalities: Facets for the n-mixing set

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ARTICLE INFO

Article history:
Received 19 September 2011
Received in revised form 10 June 2012
Accepted 3 July 2012
Available online 31 July 2012

Keywords:
Mixed n-step MIR
Mixing
Mixed integer programming
Cutting planes
Multi-module capacitated lot-sizing
Multi-module capacitated facility location

ABSTRACT

Günlük and Pochet [O. Günlük, Y. Pochet, Mixing mixed integer inequalities. Mathematical Programming 90 (2001) 429–457] proposed a procedure to *mix* mixed integer rounding (MIR) inequalities. The mixed MIR inequalities define the convex hull of the mixing set $\{(y^1, \ldots, y^m, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1 y^i + v \geq \beta_i, i = 1, \ldots, m\}$ and can also be used to generate valid inequalities for general as well as several special mixed integer programs (MIPs). In another direction, Kianfar and Fathi [K. Kianfar, Y. Fathi, Generalized mixed integer rounding inequalities: facets for infinite group polyhedra. Mathematical Programming 120 (2009) 313–346] introduced the n-step MIR inequalities for the mixed integer knapsack set through a generalization of MIR. In this paper, we generalize the mixing procedure to the n-step MIR inequalities and introduce the mixed n-step MIR inequalities. We prove that these inequalities define facets for a generalization of the mixing set with n integer variables in each row (which we refer to as the n-mixing set), i.e. $\{(y^1, \ldots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{j=1}^n \alpha_j y_j^i + v \geq \beta_i, i = 1, \ldots, m\}$. The mixed MIR inequalities are simply the special case of $n = 1$. We also show that mixed n-step MIR can generate valid inequalities based on multiple constraints for general MIPs. Moreover, we introduce generalizations of the capacitated lot-sizing and facility location problems, which we refer to as the multi-module problems, and show that mixed n-step MIR can be used to generate valid inequalities for these generalizations. Our computational results on small MIPLIB instances as well as a set of multi-module lot-sizing instances justify the effectiveness of the mixed n-step MIR inequalities.

1. Introduction

Understanding the polyhedral structures of simple mixed integer sets and using them in developing valid inequalities for general mixed integer programs (MIPs) have been a successful approach. In this paper, we consider a generalization of the well-known mixing set [1], which we refer to as the *n-mixing set*. This set is defined as follows:

$$Q^{m,n} = \left\{ (y^1, \ldots, y^m, v) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{j=1}^{n} \alpha_j y_j^i + v \geq \beta_i, i = 1, \ldots, m \right\},$$

where $\alpha_j \in \mathbb{R}, \alpha_j > 0, j = 1, \ldots, n$ and $\beta_i \in \mathbb{R}, i = 1, \ldots, m$. The mixing set studied by Günlük and Pochet [1] is the special case of $Q^{m,1}$. They showed that the mixed integer rounding (MIR) inequalities [2,3] (called *1-step* MIR inequalities in this paper) based on individual constraints of $Q^{m,1}$ can be *mixed* in a particular way to generate valid inequalities for $Q^{m,1}$, which also define the convex hull of this set. The mixed 1-step MIR inequalities can also be used to generate valid inequalities for

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doi:10.1016/j.disopt.2012.07.003

general mixed integer sets. Moreover, they generate valid inequalities for special structure MIPs such as constant-capacity lot-sizing, facility location, and network design problems [1]. Variations of the mixing set $Q^{m,1}$ have also been studied: an important variation is the mixing set with different capacities, i.e. the set

$$\widetilde{Q}^m = \left\{(y^1, \ldots, y^m, v) \in \mathbb{Z}^m \times \mathbb{R}_+ : \alpha_1^i y^i + v \geq \beta_i, i = 1, \ldots, m\right\},$$

where $\alpha_1^i, \beta_i \in \mathbb{R}$ and $\alpha_1^i > 0, i = 1, \ldots, m$. The set $\widetilde{Q}^2$ with divisible capacities, i.e. when $\alpha_1^2 | \alpha_1^1$, was studied in [4], and the set $\widetilde{Q}^2$ where capacities are not necessarily divisible was studied in [5]. The set $\widetilde{Q}^m$ with divisible capacities, i.e. when $\alpha_1^m | \alpha_1^{m-1} | \cdots | \alpha_1^1$, was studied in [6,7]. A simple algorithm for linear optimization over $\widetilde{Q}^m$ with divisible capacities along with a compact extended formulation for this set was devised in [8]. Other variants of the mixing set $Q^{m,1}$ include the continuous mixing set [9,10], the mixing set with flows [11] and the mixing set linked by bidirected paths [12]. The mixing inequalities of [1] for general mixed integer sets were studied from a group-theoretic perspective in [13] and bounds on their MIR rank were proposed in [14,15].

In another direction, Kianfar and Fathi [16] presented the n-step MIR inequalities for the general mixed integer knapsack set through a generalization of MIR. These inequalities are facet-defining for the mixed integer knapsack set under certain conditions [17]. Although their theoretical derivation is rather involved, the n-step MIR inequalities are easily generated by applying the so-called n-step MIR functions on a general mixed integer constraint. The n-step MIR functions also define extreme inequalities for the infinite group polyhedra and can be used to generate facets for finite cyclic group polyhedra [18]. A variant of the n-step MIR inequalities are the n-step mingling inequalities, which utilize the bounds on integer variables to generate stronger inequalities for general MIPs, which are facet-defining in many cases [17].

In this paper, we show that the idea of mixing can be generalized to n-step MIR inequalities. Through this generalization, we develop the *type I and type II mixed n-step MIR inequalities* for the n-mixing set $Q^{m,n}$ under the condition that for each constraint i of $Q^{m,n}$ used in the mixing, α_j's and β_i satisfy the same conditions required for validity of the n-step MIR inequality, i.e. $\alpha_j \left\lceil \beta_i^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}, j = 2, \ldots, n$ (Section 3). The mixed MIR inequalities of [1] simply correspond to the special case of $n = 1$. We then demonstrate the strength of the mixed n-step MIR inequalities by showing that the type I mixed n-step MIR inequalities define facets for the convex hull of $Q^{m,n}$, denoted by $conv(Q^{m,n})$, and type II mixed n-step MIR inequalities define faces of dimension at least $n(m-1)$ for $conv(Q^{m,n})$ and are facet-defining for this set if some additional conditions are satisfied (Section 4).

We then show how the mixed n-step MIR inequalities for $Q^{m,n}$ can be used to generate mixed n-step MIR inequalities for the general multi-constraint mixed integer set

$$Y_m = \left\{(x_1, \ldots, x_N, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+^m : \sum_{t \in T} a_{it} x_t + s_i \geq b_i, i = 1, \ldots, m\right\},$$

where $T = \{1, \ldots, N\}$ and $a_{it}, b_i \in \mathbb{R}$ for all i and t (Section 5). Note that any set defined by m mixed integer constraints can be relaxed to a set of the form Y_m (see Section 5). As a result, for a general MIP, the mixed n-step MIR generates valid inequalities that are based on multiple constraints. A mixed n-step MIR inequality for Y_m has n positive parameters, namely $\alpha_1, \ldots, \alpha_n$, which must satisfy the n-step MIR conditions, i.e. $\alpha_j \left\lceil b_i^{(j-1)}/\alpha_j \right\rceil \leq \alpha_{j-1}, j = 2, \ldots, n$, for any constraint i of Y_m that is used in generating the inequality. Any set of values for the parameters $\alpha_1, \ldots, \alpha_n$ that satisfy these conditions give a corresponding mixed n-step MIR inequality for Y_m. Notice that for the validity of the mixed n-step MIR inequality for Y_m, no conditions on the coefficients a_{it} in Y_m are required. In other words, the restriction of n-step MIR conditions is only on the parameters of the cut, i.e. $\alpha_1, \ldots, \alpha_n$, and as we will see in Section 5, there are always infinitely many choices for these parameters that satisfy the n-step MIR conditions.

Next, we introduce a generalization of the capacitated lot-sizing problem, which we refer to as the multi-module lot-sizing problem (MML), and show that the mixed n-step MIR inequalities can be used to generate valid inequalities for this problem. In MML, the total capacity in each period is the summation of integer multiples of several modules of different capacities. The mixed n-step MIR inequalities for MML generalize the (k, l, S, I) inequalities for the constant-capacity lot-sizing problem (CCL) [1,19]. Similarly, we also introduce a generalization of the capacitated facility location problem, which we refer to as the multi-module facility location problem (MMF), and show that the mixed n-step MIR inequalities can be used to generate valid inequalities for this problem. The mixed n-step MIR inequalities for MMF generalize the mixed MIR inequalities for the constant-capacity facility location problem (CCF) [1,20,21] (Section 6).

Finally, we provide our preliminary computational results on using the mixed n-step MIR inequalities in solving small MIPLIB instances [22] as well as a set of MML instances (Section 7). These results justify the effectiveness of the mixed n-step MIR inequalities.

We also note that in the special case where the parameters $\alpha_j, j = 1, \ldots, n$, in $Q^{m,n}$ are *divisible*, i.e. $\alpha_n | \alpha_{n-1} | \cdots | \alpha_1$, the n-step MIR validity conditions are always satisfied. Consequently, all results in this paper are always true for the special case of divisible parameters (as we will see in Section 6, in the case of MML and MMF, the parameters $\alpha_j, j = 1, \ldots, n$, are the capacities of modules).

First, we briefly review the necessary concepts related to the mixed MIR and the n-step MIR inequalities in Section 2.