Discrete Optimization 14 (2014) 17–33

Contents lists available at ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

Relay placement for two-connectivity☆

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ARTICLE INFO

Article history:
Received 1 August 2012
Received in revised form 8 February 2014
Accepted 17 June 2014
Available online 10 July 2014

Keywords:
Approximation algorithm
Wireless network
Steiner points
Submodular flows
Two connectivity
Parsimony

ABSTRACT

Motivated by applications to wireless sensor networks, we study the following problem. We are given a set S of wireless sensor nodes, given as a multiset of points in a normed space. We seek to place a minimum-size (multi)set Q of wireless relay nodes in the normed space such that the unit-disk graph induced by $Q \cup S$ is two-connected. The unit-disk graph of a set of points has an edge between two points if their distance is at most 1.

In Infocom 2006, Kashyap, Khuller, and Shayman present two algorithms, for the two variants of the problem: two-edge-connectivity and biconnectivity. For both they prove an approximation ratio of $2d_{MST}$, where d_{MST} is the maximum degree of a minimum-degree Minimum Spanning Tree in the normed space. It is known that in the Euclidean two-dimensional space, $d_{MST} = 5$, and in the three dimensional space, $d_{MST} = 12$.

We give a tight analysis of variants of the same algorithms, obtaining approximation ratios of d_{MST} for biconnectivity and $2d_{MST} - 1$ for two-edge-connectivity respectively. To do so we prove additional structural properties regarding bypassing Steiner nodes in biconnected graphs.

1. Introduction

A wireless sensor network is composed of a large number of sensors, which can be densely deployed to monitor the targeted environment. Some of the most important application areas of sensor networks include military, natural calamities such as forest fire detection and tornado motion, and different sorts of surveillance. When compared to traditional ad hoc networks, the most noticeable point about sensor networks is that they are limited in power, computational capacities, and memory.

Sensors may have a short transmission range, since long transmission consumes more energy, and the sensors normally have limited power. Therefore, network partitions may occur or more sensors must be placed to maintain connectivity. Higher connectivity may be desired to ensure fault-tolerance.

Formally, in the Two-Connected Relay Placement problem, we are given a set S of wireless sensor nodes, given as a multiset of points in a finite-dimensional normed space ("multiset" meaning two nodes may be placed at the same location). We seek to place a minimum-size (multi)set Q of wireless relay nodes in the normed space such that the unit-disk graph induced by $Q \cup S$ is two-connected.

A normed space is a metric space (X, d), given by a set X (of points) and a symmetric function (distance) $d : X \times X \to \mathbb{R}^+$ that obeys the triangle inequality: $\forall x, y, z \in X$, $d(x, y) \le d(x, z) + d(z, y)$, and the property that $d(x, y) = 0$ if and only if $x = y$. As defined in the literature [1], a normed space also has the following property (and others that we do not use): $\forall x, y \in X$ and $\forall \alpha \in [0, 1]$, there exists $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$ and $d(x, z) = \alpha \cdot d(x, y)$. In other words, the normed space contains all the *Steiner* points. Normed spaces of interest to wireless networks are the two and three dimensional Euclidean space, with d being the Euclidean distance (the l_2 norm).

☆ An extended abstract appeared in Proc. IFIP Networking 2012.
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http://dx.doi.org/10.1016/j.disopt.2014.06.002

A multiset allows several nodes to be placed at the same location. The unit-disk graph of a set of points has an edge between two points if their distance is at most 1 (we normalize to 1 the transmission range of the sensors). For a multiset of points Z, let $U(Z)$ be the unit-disk graph induced by Z. Also, we call two vertices U-adjacent, or U-neighbors, if their distance is at most 1.

Kashyap, Khuller, and Shayman [2,3] introduce the two variants of this problem: Two-Edge-Connected Relay Placement (when $U(S \cup Q)$ must be two-edge-connected, that is, have between any two vertices two edge-disjoint paths) and Biconnected Relay Placement ($U(S \cup Q)$ must be biconnected, that is, have between any two vertices two internally vertex-disjoint paths). Two paths are internally vertex-disjoint if they only have the endpoints in common. Biconnectivity also goes by the name of two-vertex-connectivity, or two-connectivity.

Let d_{MST} be the maximum degree of a minimum-degree Minimum Spanning Tree in the normed space. It is known [4,5] that d_{MST} is the strict Hadwiger number of the unit ball in the normed space, defined as follows: the maximum size of an independent set in $U(N_x)$, taken over all points x of the space, with N_x being the points, other than x, within distance 1 of x. It is known that $d_{MST} = 5$ in the Euclidean two-dimensional space, and $d_{MST} = 12$ in three dimensions [5].

[2] presents two algorithms, based on the Khuller and Vishkin [6] (**Algorithm KV**) and the Khuller and Raghavachari [7] (**Algorithm KR**) algorithms for Minimum-Weight Spanning Two-Edge-Connected Subgraph, and Minimum-Weight Spanning Biconnected Subgraph, respectively. For these problems, a weighted graph $G = (V, E, w)$ is given as an input, and one must select a minimum weight set of edges F such that (V, F) is two-edge-connected, or biconnected respectively. For Two-Connected Relay Placement, [2] proves that variants of the two algorithms have each approximation ratio of $2d_{MST}$.

We give a tight analysis of variants of the same algorithms, obtaining approximation ratios of d_{MST} for biconnectivity and $2d_{MST} - 1$ for two-edge-connectivity respectively. Thus, in the two-dimensional Euclidean plane, we get a ratio of 9, instead of 10 [2], for two-edge-connectivity and 5, instead of 10 [2], for biconnectivity. Assuming that no post-processing removes redundant relay nodes, the ratios given in this paragraph are tight for these algorithms.

For the ratio of $2d_{MST} - 1$, we use a more careful accounting and look inside **Algorithm KV**. For the ratio of d_{MST} for biconnectivity, we look inside **Algorithm KR**, and prove a property of biconnected graphs that may be of independent interest.

This property is technical and we only describe here a simpler version that is not used in proving our main results. We prove the following result, new to the best of our knowledge. Let H be a biconnected planar undirected graph, and replace every edge by two anti-parallel directed arcs. Let R be a subset of $V(H)$. Then there exists a set of arc-disjoint directed paths P_i of H, all starting and ending at a vertex of R and without interior vertices from R, such that, if we replace each P_i by an arc e_i joining the start and the end vertex of P_i, we obtain a biconnected digraph on R. This property allows one to "bypass" Steiner vertices ("parsimony") and in some sense eliminate them. This parsimony differs from the classical concept as given in [8] since it applies to combinatorial (and not linear programming) solutions.

For graphs in general, we prove a "fractional outconnected" variant of the property above, and use it together with **Algorithm KR** to obtain an approximation ratio of d_{MST} for biconnectivity in arbitrary normed spaces. Structural properties of biconnected Steiner networks were also studied by [9–12], and we use some of their results and techniques for our "outconnected parsimony". Using these structural properties, we construct from the optimum solution of an arbitrary Biconnected Relay Placement instance a fractional solution to a certain polytope. This polytope was proposed by Frank and Tardos [13], who proved that it is integral (see also [14]). Thus, there exists an integral solution with cost at most this fractional solution, for any non-negative cost function. We define costs that relate the objective function to an optimum relay solution, and notice that the output of **Algorithm KR** in a weighted graph that we describe later is (almost) derived from an integral polytope optimum solution.

As an example of its possible applications, this new fractional outconnected parsimony property can be used to prove that a variant of **Algorithm KR** has approximation ratio of 2 for the following network design problem: we are given a normed space and a set of terminals S. We must choose a set Z of points and a set of edges F of minimum total distance such that the graph $(Z \cup S, F)$ is biconnected. This would not be an improvement, since an approximation algorithm with a ratio of 2 is already known in finite graphs (even without a metric cost function) from the paper by Fleischer, Jain, and Williamson [15]. In Euclidean spaces, a PTAS (for any $\epsilon > 0$, there is an algorithm with approximation ratio of $1 + \epsilon$; the running time being polynomial for any *fixed* ϵ) was announced by Czumaj and Lingas [16]; their algorithms have running time exponential in the dimension and in $1/\epsilon$. Also, no fully PTAS exists [12].

In other previous works, Wang, Thai, and Du [17] and Bredin, Demaine, Hajiaghayi, and Rus [18] also gave constant factor algorithms, those of [18] achieving an $O(k^4 \log k)$ approximation for k-connectivity of $U(S \cup Q)$ in the Euclidean plane (it seems from their proof that a d_{MST} factor would apply in other normed spaces). We remark that in a wireless setting, one only needs k-connectivity between the vertices of S, i.e. k edge-disjoint (or internally vertex-disjoint) paths between any two vertices of S. For $k = 1$ or $k = 2$, by eliminating redundancy from any solution, one can see that k-connectivity or k-edge-connectivity between the vertices of S implies k-connectivity or k-edge-connectivity, respectively, of $U(S \cup Q)$. We only present the argument for 2-connectivity: If $U(S \cup Q)$ is not biconnected, it has a vertex v such that $U((S \cup Q) \setminus \{v\})$ has at least two connected components, and one of these two components contains no vertex of S, since we have two-connectivity between the vertices of S; the removal of this component does not decrease the connectivity between the vertices of S. This argument fails for $k > 2$. When requiring only k-connectivity between the vertices of S, the best known approximation ratio is obtained by Kamma and Nutov [19]: $O(d_{MST} k^2 \log k)$.