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# Connected matchings in chordal bipartite graphs

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ABSTRACT

A connected matching in a graph is a collection of edges that are pairwise disjoint but joined by another edge of the graph. Motivated by applications to Hadwiger's conjecture, Plummer, Stiebitz, and Toft (2003) introduced connected matchings and proved that, given a positive integer $k$, determining whether a graph has a connected matching of size at least $k$ is NP-complete. Cameron (2003) proved that this problem remains NP-complete on bipartite graphs, but can be solved in polynomial-time on chordal graphs. We present a polynomial-time algorithm that finds a maximum connected matching in a chordal bipartite graph. This includes a novel edge-without-vertex-elimination ordering of independent interest. We give several applications of the algorithm, including computing the Hadwiger number of a chordal bipartite graph, solving the unit-time bipartite margin-shop scheduling problem in the case in which the bipartite complement of the precedence graph is chordal bipartite, and determining – in a totally balanced binary matrix – the largest size of a square sub-matrix that is permutation equivalent to a matrix with all zero entries above the main diagonal.



## 1. Introduction

Motivated by applications to Hadwiger's conjecture, Plummer, Stiebitz, and Toft [1] introduced connected matchings and exhibited, for graphs with independence number two, a close connection between the maximum size of a connected matching and the Hadwiger number of a graph. They also proved that, given a positive integer $k$, determining whether a graph has a connected matching of size at least $k$ is NP-complete. This they accomplish by reducing the (well known NP-complete) $k$-clique problem to the problem of determining whether a graph has a connected matching of size at least $k$. Cameron [2] proved that the problem of determining whether a graph has a connected matching of size $k$ remains NP-complete on bipartite graphs, but can be solved in polynomial-time on chordal graphs.

We present a polynomial-time algorithm that finds a maximum connected matching in a chordal bipartite graph. Note that chordal bipartite graphs are not necessarily chordal graphs, hence Cameron's polynomial-time algorithm for chordal graphs does not imply that the maximum connected matching problem can be solved in polynomial-time on chordal bipartite graphs. Cameron's algorithm combines two important observations: chordal graphs have a polynomial number of maximal cliques and, a maximum-sized connected matching in a chordal graph necessarily consists of pairwise disjoint edges all incident to one maximal clique. Analogues of these observations exist for chordal bipartite graphs: chordal bipartite graphs have a polynomial number of maximal bicliques (see bewvo's in Section 4) and, a maximum-sized connected matching in a chordal bipartite graph necessarily consists of pairwise disjoint edges all incident to one biclique (see

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Theorem 17 in Section 7). Unfortunately these analogues fail to yield an analogue to Cameron's algorithm because pairwise disjoint edges incident to a maximal biclique need not be connected, and the biclique incident to all edges of a maximum connected matching need not be maximal. Instead our algorithm employs a common strategy for chordal bipartite graphs: dynamic programming. We develop and apply a novel edge-without-vertex-elimination ordering of independent interest. Using these tools rather than the aforementioned analogues makes our algorithm similar in spirit (not complexity) to the maximum induced matching algorithm developed by Brandstädt and Hoàng [3].

The maximum connected matching problem has received increasing independent attention in the literature, particularly because it is a natural variation of popularly investigated matching variations in bipartite graphs (e.g., the maximum induced matching problem [3], the cross-free matching problem [4]).

In the next section we present common definitions and notation. The following section reduces the maximum connected matching problem for chordal bipartite graphs to the problem of computing supports (defined in Section 3) of maximal bicliques. Section 4 introduces a novel 'diverse' basic edge ordering designed to drive the dynamic programming algorithm (presented in Section 5) that computes the supports of the maximal bicliques of a chordal bipartite graph.

In Sections 6 and 7 we give several applications of our algorithm, including computing the Hadwiger number of a chordal bipartite graph, solving the unit-time bipartite margin-shop scheduling problem in the case in which the bipartite complement of the precedence graph is chordal bipartite, and determining – in a totally balanced binary matrix – the largest size of a square sub-matrix that is permutation equivalent to a matrix with all zero entries above the main diagonal.

## 2. Preliminaries

All graphs in this paper are finite, simple and undirected. A graph is *trivial* if it has no edges; otherwise it is *nontrivial*. Edges are unordered pairs of vertices, but following standard notation, the edge $e = \{a, b\}$ is abbreviated $ab$. The vertices $a$ and $b$ are the *endpoints* of the edge $e = ab$. The edge $ab$ is *incident* to $a$ and $b$. The *neighbors* of a vertex $u$ in the graph $G$ is the set $N(u) = \{v \in V : uv \in E\}$, which is sometimes denoted as $N_G(u)$ to emphasize the graph. A vertex $v$ is *isolated* in $G$ if $N_G(v) = \varnothing$; otherwise it is *non-isolated*. An *independent set* is a collection of pairwise non-adjacent vertices. A *bipartite graph* is a graph that admits a vertex partition into at most two independent sets called the *partite sets*. A bipartite graph $G$ with edge set $E$ and partite sets $A$ and $B$ is denoted as $G(A, B; E)$; with this notation it is implicit that $V = A \cup B$ is the vertex set of $G$. A bipartite graph $G(A, B; E)$ is *standard* if $|A| \leq |B|$. A bipartite graph is *chordal bipartite* if it has no chordless cycles of length greater than 4. A *matching* is a collection of disjoint edges. An edge that intersects two disjoint edges is said to *join* them; that is, edge $g$ joins edges $e$ and $f$ if $e \cap f = \varnothing$ and $e \cap g \neq \varnothing \neq f \cap g$. A matching $M$ in a graph $G$ is *connected* if every pair of distinct edges of $M$ is joined by at least one edge of $G$. The maximum size of a connected matching in a graph $G$ is denoted by $\nu_c(G)$. A matching *saturates* a vertex $v$ if there is an edge of the matching incident to $v$. Similarly, a matching *saturates* a subset of the vertices if it saturates each vertex in the subset.

A subset of vertices is called a *biclique* if it induces a complete bipartite graph. An edge $e = ab$ is *bisimplicial* if $N(a) \cup N(b)$ is a biclique. If $e = ab$ is a bisimplicial edge, $Q_e = N(a) \cup N(b)$ denotes the associated biclique. A vertex is *weakly simplicial* if its neighbors form an independent set and the neighbors can be ordered $v_1, v_2, \ldots, v_k$ so that

$$N(v_1) \subseteq N(v_2) \subseteq \cdots \subseteq N(v_k).$$

Observe that an isolated vertex is vacuously weakly simplicial. A neighbor $v$ of a weakly simplicial vertex $u$ is called an *associate* of $u$ if $N(v) \subseteq N(v')$, for all $v' \in N(u)$. An edge $ab$ in the standard chordal bipartite graph (with $a \in A$ and $b \in B$) is *basic* if $a$ is a weakly simplicial vertex and $b$ is an associate of $a$.

We shall apply the following facts. Facts 1 and 2 are straightforward observations.

**Fact 1.** *Basic edges are bisimplicial.*

**Fact 2.** *If $G$ is a chordal bipartite graph and $e$ is a bisimplicial edge of $G$, then $G - e$ is also chordal bipartite.*

A graph on four vertices and two disjoint edges is denoted as $2K_2$. A $2K_2$-*free graph* is a graph containing no vertex-induced subgraph isomorphic to $2K_2$. A bipartite, $2K_2$-free graph is called a *chain graph* (also known as a difference graph). Yannakakis [5] proved that a connected bipartite graph is $2K_2$-free if and only if the vertices of one partite set can be linearly ordered by their neighborhoods; that is, the neighborhoods form a chain under the inclusion ordering.

**Fact 3.** *If $G(A, B; E)$ is a chordal bipartite graph, then $a \in A$ is weakly simplicial if and only if the subgraph of $G$ induced by $A \cup N(a)$ is $2K_2$-free.*

**Fact 4** (*See Corollary 4.6, [6]*)**.** *A graph is chordal bipartite if and only if every induced subgraph has a weakly simplicial vertex. Furthermore, a nontrivial chordal bipartite graph has a weakly simplicial vertex in each partite set and at least two weakly simplicial vertices in each partite set that contains at least two vertices.*