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A new separation algorithm for the Boolean quadric and cut polytopes

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ABSTRACT

A *separation algorithm* is a procedure for generating cutting planes. Up to now, only a few polynomial-time separation algorithms were known for the *Boolean quadric* and *cut* polytopes. These polytopes arise in connection with zero–one quadratic programming and the max-cut problem, respectively. We present a new algorithm, which separates over a class of valid inequalities that includes all odd bicycle wheel inequalities and $(2p+1, 2)$-circulant inequalities. It exploits, in a non-trivial way, three known results in the literature: one on the separation of $\{0, \frac{1}{2}\}$-cuts, one on the symmetries of the polytopes in question, and one on an affine mapping between the polytopes.

1. Introduction

A popular way to tackle hard combinatorial optimisation problems is to formulate them as Integer Linear Programs (ILPs), define an associated family of polytopes, and then derive linear inequalities that define faces (preferably facets) of those polytopes (see, e.g., [1,2]). These inequalities can then be used as cutting planes within a branch-and-cut framework (see, e.g., [2–4]).

In order actually to use a class of inequalities as cutting planes, one needs a *separation* algorithm. A separation algorithm, for a given family of polytopes and a given class of inequalities, is an algorithm that takes as input a point that does not lie in one of the polytopes, and outputs a violated inequality in the given class, if one exists [5]. A great deal of the research on separation algorithms has been carried out in the context of the *traveling salesman problem* (see, e.g., [6,7]). Even so, useful separation algorithms have been discovered for many other $\mathcal{NP}$-hard combinatorial optimisation problems; see [1–4] for surveys.

In this paper, we are concerned with the so-called *Boolean quadric* and *cut* polytopes. The Boolean quadric polytope, first defined by Padberg [8], arises in the context of *unconstrained zero–one quadratic programming*. The cut polytope, defined by Barahona and Mahjoub [9], arises in connection with the *max-cut* problem. Both problems have a wide array of important applications (see, e.g., [10,11]). The reason that we consider these polytopes together is that there is a well-known affine mapping from one to the other, known as the *covariance map* [8,11–13].

A vast array of valid and facet-defining inequalities have been discovered for the Boolean quadric and cut polytopes (see the survey in [11]). On the other hand, there exist relatively few separation algorithms (see the next section). In this paper, we present a new separation algorithm, and show that it separates over a class of inequalities that includes all of the so-called *odd bicycle wheel* inequalities [9] and $(2p+1, 2)$-*circulant* inequalities [14]. A separation algorithm with this property

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was already found by one of the authors [15], but it was impractical, being based on solving a series of large linear programs. Our new algorithm is much faster.

We remark that our algorithm exploits, in a non-trivial way, three known results in the literature:

1. A result, due to Caprara and Fischetti [16], on the complexity of separation for a class of cutting planes for general ILPs, called $\{0, \frac{1}{2}\}$-cuts.
2. A result, due to Barahona and Mahjoub [9], about the invariance of the cut polytope with respect to the so-called *switching* operation.
3. The result mentioned above, about the equivalence of the Boolean quadric and cut polytopes under the covariance map.

The structure of the paper is as follows. Section 2 is the literature review. In Section 3, we present some simple valid inequalities for the Boolean quadric polytope. These will be used later, in our separation algorithm, to generate more complex inequalities. The separation algorithm itself is presented in Section 4, along with the analysis of its running time. Section 5 presents the algorithm for the cut polytope, and then shows how it can be used to derive a second separation algorithm for the Boolean quadric polytope, which is slower, but separates over a wider class of inequalities. Finally, some concluding remarks are made in Section 6.

Throughout the paper, we let V_n and E_n denote the vertex and edge sets, respectively, of a complete undirected graph of order n. That is, V_n denotes $\{1, \ldots, n\}$ and E_n denotes $\{S \subset V_n : |S| = 2\}$.

2. Literature review

In this section, we review the relevant literature. We cover the polytopes in Section 2.1, the covariance map and switching in Section 2.2, valid inequalities in Section 2.3, separation routines in Section 2.4, and $\{0, \frac{1}{2}\}$-cuts in Section 2.5.

2.1. The polytopes

The *Boolean quadric polytope* of order n, denoted by BQP_n, is the convex hull of vectors $(x, y) \in \{0, 1\}^{V_n + E_n}$ satisfying $y_{ij} = x_i x_j$ for all $\{i, j\} \in E_n$ (Padberg [8]). A vector $(x, y) \in \mathbb{Z}^{V_n+E_n}$ is an extreme point of BQP_n if and only if it satisfies the following linear inequalities, due to Fortet [17]:

$$y_{ij} \geq 0 \quad (\{i, j\} \in E_n) \tag{1}$$

$$y_{ij} - x_i \leq 0 \quad (i \in V_n,\ j \in V_n \setminus \{i\}) \tag{2}$$

$$x_i + x_j - y_{ij} \leq 1 \quad (\{i, j\} \in E_n). \tag{3}$$

We will call these *trivial* inequalities.

Given any vertex set $S \subseteq V_n$, the edge-set

$$\delta(S) = \{\{i, j\} \in E_n : i \in S, j \in V_n \setminus S\}$$

is called a *cut*. The *cut polytope* of order n, denoted by CUT_n, is the convex hull of vectors $z \in \{0, 1\}^{E_n}$ that are incidence vectors of cuts (Barahona and Mahjoub [9]). As noted in [9], CUT_n is the convex hull of vectors $z \in \{0, 1\}^{E_n}$ satisfying the following linear inequalities:

$$z_{ij} - z_{ik} - z_{jk} \leq 0 \quad (\{i, j\} \in E_n,\ k \in V_n \setminus \{i, j\}) \tag{4}$$

$$z_{ij} + z_{ik} + z_{jk} \leq 2 \quad (\{i, j, k\} \subset V_n). \tag{5}$$

These are called *triangle* inequalities [9].

2.2. The covariance map and switching

It was pointed out in [8,12,13] that a point (x^*, y^*) belongs to BQP_n if and only if the point z^* belongs to CUT_{n+1}, where:

$$z^*_{i,n+1} = x^*_i \quad (i \in V_n)$$
$$z^*_{ij} = x^*_i + x^*_j - 2y^*_{ij} \quad (\{i, j\} \in E_n).$$

This linear mapping is called the *covariance map*. A consequence of this map is that the inequality $\alpha^T z \leq \beta$ is valid for CUT_{n+1} if and only if the inequality

$$\sum_{i \in V_n} \left(\sum_{j \in V_{n+1} \setminus \{i\}} \alpha_{ij} \right) x_i - 2 \sum_{e \in E_n} \alpha_e y_e \leq \beta$$

is valid for BQP_n.