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A branch-and-cut algorithm for the capacitated profitable tour problem

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ABSTRACT

This paper considers the *Capacitated Profitable Tour Problem* (CPTP) which is a special case of the *Elementary Shortest Path Problem with Resource Constraints* (ESPPRC). The CPTP belongs to the group of problems known as traveling salesman problems with profits. In CPTP each customer is associated with a profit and a demand and the objective is to find a capacitated tour (rooted in a depot node) that minimizes the total travel distance minus the profit of the visited customers. The CPTP can be recognized as the sub-problem in many column generation applications, where it is traditionally solved through dynamic programming. In this paper we present an alternative framework based on a formulation for the undirected CPTP and solved through branch-and-cut. Valid inequalities are presented among which we introduce a new family of inequalities for the CPTP denoted *rounded multistar inequalities* and we prove their validity. Computational experiments are performed on a set of instances known from the literature and a set of newly generated instances. The results indicate that the presented algorithm is highly competitive with the dynamic programming algorithms. In particular, we are able to solve instances with 800 nodes to optimality where the dynamic programming algorithms cannot solve instances with more than 200 nodes. Moreover dynamic programming and branch-and-cut complement each other well, giving us hope for solving more general problems through hybrid approaches. The paper is intended to serve as a platform for further development of branch-and-cut algorithms for CPTP hence also acting as a survey/tutorial.

1. Introduction

The *capacitated profitable tour problem* (CPTP) can be defined on a complete undirected graph $G(V, E)$ with nodes $V = N \cup \{0\}$ where N is a set of customers and 0 is the depot node, and E is the set of edges connecting the nodes in V. A cost c_e is associated with each edge $e \in E$. Also, a demand d_i and a profit p_i are associated with each customer $i \in N$ and a capacity Q is given for the maximum load of tour. The objective is to find a tour rooted in the depot where the demand accumulated at the customers does not exceed the capacity, and the total travel distance minus the profits gained by visiting customers is minimized. The problem is a special case of the *elementary shortest path problem with resource constraints* (ESPPRC) since the CPTP can be reduced to ESPPRC, by transferring the demands from the nodes onto the arcs. Notice that there is no way to transfer resource-consumption from arcs to nodes, hence the ESPPRC cannot be reduced to the CPTP.

The CPTP is a side-constrained version of the profitable tour problem named by Dell'Amico et al. [1], a problem that falls within the category of traveling salesman problems with profits as classified by Feillet et al. [2]. Other problems in this

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category are the orienteering problem (OP) (also known as the selective traveling salesman problem) and the prize-collecting traveling salesman problem (PCTSP). In the OP the total tour length is bounded from above, and the objective is to maximize the profit gained by visiting customers. In the PCTSP the objective is similar to the CPTP but a minimum amount of profits must be collected on the tour. In the context of the capacitated vehicle routing problem (CVRP) the CPTP appears as the sub-problem in column generation methods, see e.g. Baldacci et al. [3,4]. In this context, the CPTP is often transformed to a path problem (a path is obtained from the tour by splitting the depot into two nodes) and is denoted the elementary shortest path problem with resource constraints [5]. The resource is given as an accumulation of demand of the visited customers and is constrained by the capacity. However, in recent routing applications the sub-problem is complicated considerably by the introduction of additional cuts in the column generating master problem, such as the strong capacity inequalities [4], the subset-row inequalities [6], the Chvátal–Gomory rank-1 cuts [7], and the clique inequalities [8]. The sub-problem can no longer be considered a CPTP. Moreover, the sub-problems are often solved as feasibility problems instead of optimization problems which may favor other types of combinatorial algorithms than branch-and-cut algorithms.

Laporte and Martello [9] showed that the OP is $\mathcal{NP}$-hard by reduction from the Hamiltonian circuit problem. Using a similar reduction it can be shown that the CPTP also belongs to the class of $\mathcal{NP}$-hard problems. If there are no cycles with negative cost in the graph G, then the CPTP is solvable in pseudo-polynomial time using a dynamic programming algorithm. In this particular case the CPTP relates to the constrained shortest path problem (again by transformation to a path problem). Several algorithms based on dynamic programming exist for this problem, see e.g., Beasley and Christofides [10], Carlyle et al. [11], Dumitrescu and Boland [12], and Muhandiramge and Boland [13].

Bixby [14], Bixby et al. [15] consider the CPTP in her Ph.D. thesis on the CVRP and present a mathematical model and a branch-and-cut (BAC) algorithm. Letchford and Salazar-Gonzalez [16] discuss projection results for the CVRP and present two families of multistar inequalities that are valid for the CPTP. Other work on the CPTP in a CVRP context is mainly concerned with dynamic programming algorithms. Feillet et al. [17] present a dynamic programming algorithm where the elementarity of the path is ensured by the use of an additional resource per node. Chabrier [18] improved on the labeling algorithm by applying various bounding and dominance procedures to avoid the extension of unpromising paths. Christofides et al. [19] proposed a bi-directional labeling algorithm where paths are extended from both ends of the path until half of the capacity is reached. The partial paths are then combined to construct a full path. Righini and Salani [20] generalized this approach to other types of resources. Independently, Boland et al. [21] and Righini and Salani [22] proposed to initially relax the node resources and add them iteratively until the path is elementary. In the former paper this is referred to as a *state space augmentation* algorithm and in the latter it is denoted a *decremental state space relaxation* algorithm. Furthermore, Righini and Salani [22] propose to use the result of the relaxed problem in a branch-and-bound algorithm. Fischetti et al. [23] and Gendreau et al. [24] present BAC algorithms for the OP. They present several valid inequalities, many of which are also valid for the CPTP. Indeed, we prove that the polytope of the CPTP can be transformed to an instance of the polytope for the OP. However, Gendreau et al. [24] also present some inequalities related to the objective function of the OP that are not valid for the CPTP. Bauer et al. [25] consider the cardinality constrained circuit problem (CCCP) where a minimum cost circuit of maximal cardinality in a graph is sought. The CCCP is equivalent to the CPTP with unit demands if one node is fixed in the CCCP (the depot node of the CPTP). Two mathematical models are presented and several valid inequalities are investigated. Bauer et al. [25] suggest to solve the CPTP by a BAC algorithm, but to our knowledge this has not been pursued.

The contribution of this paper is the introduction of an IP model for the CPTP and a BAC algorithm for solving it. This includes the adaption of several valid inequalities from e.g. the OP and the CCCP, the introduction of the *rounded multistar inequalities*, and a proof of validity for the inequalities with regard to the CPTP. Also, we have successfully implemented a separation heuristic for finding *knapsack large multistar inequalities*, that prove their usefulness for the CPTP. The computational experiments show that the separation algorithms are able to find *knapsack large multistar* inequalities in many benchmark sets, where previous authors were much less successful, and several *rounded capacity* and *generalized large multistar* inequalities. Computational experiments on a set of instances known from the literature and a set of newly generated instances show that the BAC algorithm is competitive with state-of-the-art dynamic programming algorithms. In particular, the BAC algorithm is able to solve instances with 800 nodes to optimality where the dynamic programming algorithms cannot solve instances with more than 200 nodes. In general the two algorithms appear to complement each other well. The BAC algorithm performs best on those instances that are difficult to solve by dynamic programming, hence the paper is intended to serve as a platform for further development of branch-and-cut algorithms for CPTP giving a survey/tutorial of all cuts.

The paper is organized as follows: Section 2 contains an integer programming model for the CPTP, Section 3 describes the cutting planes used in the BAC algorithm, Section 4 presents the separation results for these cutting planes, the computational results are found in Section 5, and Section 6 concludes the work.

2. Mathematical model

Recall the definition of CPTP on a graph $G(V, E)$. The traversal of an edge e is indicated by the binary variable x_e for all $e \in E$ and a visit to node i is indicated by the binary variable y_i for all $i \in V$. Some short-hand notation: for node set S we use $\delta(S)$ to indicate the edge set consisting of the edges between S and its complement $\overline{S}$, $E(S)$ to indicate the edge set of the complete sub-graph spanned by S, and $E(S : T)$ for $T \cap S = \emptyset$ to indicate the edges connecting S and T. For singleton sets we simply write i instead of $\{i\}$, e.g., $\delta(i)$ is the set of edges connected to node i.