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Note

A revised reformulation-linearization technique for the quadratic assignment problem

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ABSTRACT

The Reformulation Linearization Technique (RLT) applied to the Quadratic Assignment Problem yields mixed 0–1 programming problems whose linear relaxations provide a strong bound on the objective value. Nevertheless, in the high level RLT representations the computation requires much effort. In this paper we propose a new compact reformulation for each level of the RLT representation exploiting the structure of the problem. Computational results on some benchmark instances indicate the potential of the new RLT representations as the level of the RLT increases.

1. Introduction

Consider the general form of the Quadratic Assignment Problem (QAP) proposed by Lawler [1] as follows:

$$\text{QAP}: \min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} q_{ijkl}x_{ij}x_{kl} + \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}x_{ik} \tag{1}$$

$$\text{s.t. } x \in X,\ \ x \text{ binary} \tag{2}$$

where

$$X = \left\{ x \geq 0 : \sum_{j=1}^{n} x_{ij} = 1\ \forall i = 1,\ldots,n;\ \sum_{i=1}^{n} x_{ij} = 1\ \forall j = 1,\ldots,n \right\}. \tag{3}$$

The QAP was first introduced by Koopmans and Beckmann [2] in the context of facility location to deal with a one-to-one assignment of n facilities to n locations when the objective function accounts for interaction costs between pairs of facilities. Since then, the QAP has been used to model many applications including, among others, backboard wiring [3], typewriter keyboards and control panels design [4], scheduling [5], storage-and-retrieval [6]. It has been shown that the QAP is among the most difficult NP-hard combinatorial optimization problems and, in general, solving instances of size $n \geq 30$ in a reasonable time is impossible [7]. Due to its quadratic nature, many attempts have been made in the literature to linearize the objective function so that the resulting lower bound is strong enough to be used in a branch-and-bound algorithm. Among the best lower bounding approaches in the literature we can refer the reader to Frieze and Yadegar [8], Carraresi and Malucelli [9,10], Adams and Johnson [11], Karisch et al. [12], the level-1 RLT dual-ascent bound by Hahn and

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Grant [13], the convex quadratic programming bound by Anstreicher and Brixius [14], the level-2 RLT by Adams et al. [15], and level-3 RLT by Hahn et al. [16].

In this paper we present revised versions of the RLT representations for the QAP. The main idea is to remove some set of constraints in each level of the RLT representation of the QAP so that the resulting problem remains equivalent to the original one, and the set of new constraints possesses the block-diagonal structure.

2. Reformulation-linearization technique

In this section we present the Reformulation-linearization technique (RLT) applied to the QAP. Based on the RLT technique for general zero–one polynomial programs by Adams and Sherali [17,18], the first RLT representation for the QAP was introduced by Adams and Johnson [11]. Consider problem QAP as presented in (1)–(3). The level-1 RLT representation is generated via the following two steps:

Reformulation: Multiply each of the $2n$ equations and each of the n^2 nonnegativity constraints defining X by each of the n^2 binary variables x_{kl}, and append these new constraints to the formulation. When the variable x_{ij} in a given constraint is multiplied by x_{kl}, express the resulting product as $x_{ij}x_{kl}$ in that order. Substitute x_{kl}^2 with x_{kl} throughout the constraints and set $x_{ij}x_{kl} = 0$ if $i = k$ and $j \neq l$ or $i \neq k$ and $j = l$.

Linearization: For all (i, j, k, l) with $i \neq k$ and $j \neq l$, substitute each product $x_{ij}x_{kl}$ with y_{ijkl}. Enforce the equality $y_{ijkl} = y_{klij}$ for all (i, j, k, l) with $i < k$ and $j \neq l$.

The level-1 RLT results as follows:

$$\text{RLT1}: \min \sum_i \sum_j \sum_{k\neq i} \sum_{l\neq j} q_{ijkl} y_{ijkl} + \sum_i \sum_j c_{ij} x_{ij} \tag{4}$$

$$\text{s.t.} \sum_{i\neq k} y_{ijkl} = x_{kl} \quad \forall (j, k, l),\ j \neq l \tag{5}$$

$$\sum_{j\neq l} y_{ijkl} = x_{kl} \quad \forall (i, k, l),\ i \neq k \tag{6}$$

$$y_{ijkl} = y_{klij} \quad \forall (i, j, k, l),\ i < k,\ j \neq l \tag{7}$$

$$y_{ijkl} \geq 0 \quad \forall (i, j, k, l),\ i \neq k,\ j \neq l \tag{8}$$

$$x \in X, \quad x \text{ binary}. \tag{9}$$

Note that for any feasible solution (x, y) to RLT1, the RLT theory enforces the equations $y_{ijkl} = x_{ij}x_{kl}$ for all $(i, j, k, l),\ i \neq k,\ j \neq l$ [17,19]. Thus we have the following:

Proposition 1. *Problems QAP and RLT* 1 *are equivalent.*

Eq. (7) is very important and says that if an element y_{ijkl}, $i \neq k, j \neq l$ is part of a solution (i.e., equal to 1) then it has a "complementary element" y_{klij} that is also in that solution. In general, the RLT1 representation has a large number of variables and constraints, which makes it computationally challenging, even for small QAP instances. Resende et al. [20] performed a computational test of the lower bounds generated by the LP relaxation of the RLT1. They reduced the numbers of variables and constraints in RLT1 by removing all variables y_{ijkl} with $i > k$ and $j \neq l$ and by making the substitutions suggested by (7) throughout the objective function and constraints. Then they solved the LP relaxation by using an experimental interior point method code, called ADP. To solve the RLT1, Adams and Johnson provide a Lagrangian relaxation which has a block-diagonal structure. More precisely they dualize constraints (7) on the complementary pairs and decompose the resulting problem into n^2 separate linear assignment problems of size $n - 1$ and a linear assignment problem of size n. Hahn and Grant [13] gave a different interpretation of the same decomposition for lower bound calculation by using a dual-ascent strategy. Their dual-ascent procedure gives a bound very close to optimum of the LP relaxation of the RLT1, improving upon the computational results of Adams and Johnson [11], and requiring only a small fraction of the time of Resende et al. [20].

Based on the success of level-1 RLT representation to gain a tight bound for the QAP and also due to the block-diagonal structure of the problem which lends itself to efficient solution methods, the level-2 and level-3 RLT can be defined in the same way as the level-1 RLT via the reformulation and linearization steps. In the level-2 RLT representation, in addition to the operations done in the level-1, each binary variable in X is multiplied also by products $x_{kl}x_{pq}$ having $k \neq p$ and $l \neq q$. For more details concerning the reformulation and linearization step for the level-2 RLT we refer the reader to [15]. The level-2 RLT is called RLT2 and is written as follows:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} q_{ijkl} y_{ijkl} + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{10}$$

$$\text{s.t.} \sum_{i\neq k,p} z_{ijklpq} = y_{klpq} \quad \forall (j, k, l, p, q),\ j \neq l \neq q,\ k \neq p \tag{11}$$