Discrete Optimization 14 (2014) 111–125

Contents lists available at ScienceDirect

Discrete Optimization

journal homepage: www.elsevier.com/locate/disopt

Multi-dimensional vector assignment problems

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ARTICLE INFO

Article history:
Received 20 October 2013
Received in revised form 22 July 2014
Accepted 29 August 2014
Available online 21 September 2014

Keywords:
Multi-dimensional assignment
Approximability
Worst-case analysis
Submodularity
Wafer-to-wafer integration

ABSTRACT

We consider a special class of axial multi-dimensional assignment problems called *multi-dimensional vector assignment* (MVA) problems. An instance of the MVA problem is defined by m disjoint sets, each of which contains the same number n of p-dimensional vectors with nonnegative integral components, and a cost function defined on vectors. The cost of an m-tuple of vectors is defined as the cost of their component-wise maximum. The problem is now to partition the m sets of vectors into n m-tuples so that no two vectors from the same set are in the same m-tuple and so that the sum of the costs of the m-tuples is minimized. The main motivation comes from a yield optimization problem in semi-conductor manufacturing. We consider a particular class of polynomial-time heuristics for MVA, namely the *sequential* heuristics, and we study their approximation ratio. In particular, we show that when the cost function is monotone and subadditive, sequential heuristics have a finite approximation ratio for every fixed m. Moreover, we establish smaller approximation ratios when the cost function is submodular and, for a specific sequential heuristic, when the cost function is additive. We provide examples to illustrate the tightness of our analysis. Furthermore, we show that the MVA problem is APX-hard even for the case $m = 3$ and for binary input vectors. Finally, we show that the problem can be solved in polynomial time in the special case of binary vectors with fixed dimension p.

1. Introduction

1.1. Problem statement

We consider a multi-dimensional assignment problem motivated by an application arising in the semi-conductor industry. Formally, the input of the problem is defined by m disjoint sets $V_1, \ldots, V_m$, where each set V_k contains the same number n of p-dimensional vectors with nonnegative integral components, and by a *cost function* $c(u) : \mathbb{Z}_+^p \to \mathbb{R}_+$. Thus, the cost function assigns a nonnegative cost to each p-dimensional vector.

A (*feasible*) *m-tuple* is an mp-dimensional vector $(u^1, u^2, \ldots, u^m) \in V_1 \times V_2 \times \cdots \times V_m$. Define the component-wise maximum operator $\vee$ as follows: for every pair of vectors $u, v \in \mathbb{Z}_+^p$,

$$u \vee v = (\max(u_1, v_1), \max(u_2, v_2), \ldots, \max(u_p, v_p)).$$

We extend the definition of the cost function to k-tuples, for any $k \geq 1$, by setting $cost(u^1, \ldots, u^k) := c(u^1 \vee \cdots \vee u^k)$. More generally, when W is any set of k-tuples, we let $cost(W) = \sum_{a \in W} cost(a)$.

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http://dx.doi.org/10.1016/j.disopt.2014.08.005
1572-5286/

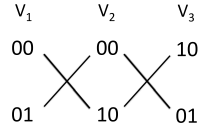

Fig. 1. A WWI-3 instance with $m = 3$, $n = p = 2$ and an illustration of the optimal solution.

A *feasible assignment* for $V = V_1 \times \cdots \times V_m$ is a set A of n feasible m-tuples such that each element of $V_1 \cup \cdots \cup V_m$ appears in exactly one m-tuple of A. In view of the previous definitions, the cost of a feasible assignment A is the sum of the costs of its m-tuples:

$$cost(A) = \sum_{(u^1,\ldots,u^m)\in A} c(u^1 \vee \cdots \vee u^m). \tag{1.1}$$

With this terminology, the *multi-dimensional vector assignment problem* (MVA-m, or MVA for short) is to find a feasible assignment for V with minimum cost. A case of special interest is the case when all vectors in $V_1 \cup \cdots \cup V_m$ are binary 0–1 vectors; we call this special case *binary* MVA. Finally, the *wafer-to-wafer integration problem* (WWI-m or WWI for short) arises when the cost function of the binary MVA is *additive*, meaning that $c(u) = \sum_{i=1}^{p} u_i$ for all $u \in \mathbb{Z}_+^p$.

In this paper, we investigate how closely the optimal solution of MVA-m and WWI-m can be approximated by a particular class of approximation algorithms.

Example 1. An instance of WWI with $m = 3$, $n = p = 2$ is displayed in Fig. 1. The optimal value of the instance is equal to 2: it is achieved by assigning the first vector of V_1, the second vector of V_2, and the first vector of V_3 to the same triple, thus arriving at vector $(1, 0)$ with cost $c(1, 0) = 1$; the remaining three vectors form a second triple with cost $c(0, 1) = 1$.

1.2. Wafer-to-wafer integration and related work

The motivation for studying the WWI problem arises from the optimization of the wafer-to-wafer production process in the electronics industry. We only provide a brief description of this application; for additional details, we refer to papers by Reda, Smith and Smith [1], Taouil and Hamdioui [2], Taouil et al. [3], and Verbree et al. [4].

For our purpose, a *wafer* can be viewed as a string of elements called *dies*. Each die can be either good (operative) or bad (defective). So, a wafer can be modeled as a binary vector, where each '0' represents a good die and each '1' represents a bad die. There are m lots of wafers, say $V_1, \ldots, V_m$, and each lot contains n wafers. All wafers in a given lot are meant to have identical functionalities, were it not for the occasional occurrence of defective dies during the previous production steps. The wafer-to-wafer integration process requires to form *stacks*, where a stack is obtained by "superposing" m wafers chosen from different lots; thus, a stack corresponds to a feasible m-tuple. As a result of integration, each position in the stack gives rise to a *three-dimensional stacked integrated circuit* (3D-SIC) which is 'good' only when the corresponding m entries of the selected wafers are 'good'; otherwise, the 3D-SIC is 'bad'. The yield optimization problem now consists in assigning the available wafers to n stacks so as to minimize the total number of bad 3D-SICs. Thus, the WWI problem provides a model for yield optimization.

The wafer-to-wafer yield optimization problem has recently been the subject of much attention in the engineering literature. One example is the contribution by Reda et al. [1]. These authors formulate WWI as a multi-dimensional assignment problem. A natural formulation of WWI as an integer linear programming problem turns out to be hard to solve to optimality for instances with large values of m (typical dimensions for the instances are: $3 \le m \le 10$, $25 \le n \le 75$, $500 \le p \le 1000$). On the other hand, Reda et al. [1] propose several heuristics and show that they perform well in computational experiments. Some recent work in this direction is also reported in [5,2–4].

Our main objective in this paper is twofold: (i) to derive approximation ratios for so-called *sequential heuristics* when they are applied to the MVA problem and to the (more specific) WWI problem, and (ii) to investigate the approximability of the WWI problem. Let us note at this point that the wafer-to-wafer integration problem is usually formulated in the literature as a maximization problem (since one wants to maximize the yield). However, we feel that from the approximation point of view, it is more appropriate to study its cost minimization version. Indeed, in industrial instances, the number of bad dies in each wafer is typically much less than the number of good dies. Therefore, it is more relevant to be able to approximate the (smaller) minimum cost than the (larger) maximum yield.

Since MVA is defined as a multi-dimensional assignment problem with a specific cost structure, our work relates to previous publications on special classes of multi-dimensional assignment problems, such as Bandelt, Crama and Spieksma [6], Burkard, Rudolf and Woeginger [7], Crama and Spieksma [8], Dokka, Kouvela and Spieksma [9], Goossens et al. [10], Spieksma and Woeginger [11], etc. Surveys on multi-dimensional assignment problems can be found in [12,13]. The composition of sum and max operators in the cost function (1.1) is superficially reminiscent of max-algebra formulations of assignment problems, such as those discussed in [12] or [14]. To the best of our knowledge, however, the approximability of MVA has only been previously investigated by Dokka et al. [15], who mostly focused on the case $m = 3$ with additive cost functions. The present paper extends to MVA-m, $m \ge 3$, and considerably strengthens the results presented in [15]. Finally, we point