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Pareto optimality in many-to-many matching problems

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ABSTRACT

Consider a many-to-many matching market that involves two finite disjoint sets, a set A of applicants and a set C of courses. Each applicant has preferences on the different sets of courses she can attend, while each course has a quota of applicants that it can admit. In this paper, we examine Pareto optimal matchings (briefly POM) in the context of such markets, that can also incorporate additional constraints, e.g., each course bearing some cost and each applicant having a limited budget available. We provide necessary and sufficient conditions for a many-to-many matching to be Pareto optimal and show that checking whether a given matching is Pareto optimal can be accomplished in $O(|A|^2 \cdot |C|^2)$ time. Moreover, we provide a generalized version of serial dictatorship, which can be used to obtain any many-to-many POM. We also study some structural questions related to POM. We show that, unlike in the one-to-one case, finding a maximum cardinality POM is NP-hard for many-to-many markets.

1. Introduction

A university runs a leisure centre that offers a variety of activities (we shall call all of them *courses*), to students and employees (we shall call all of them *applicants*). Each applicant can attend one or more courses and, because of various technical constraints, each course can only accept a restricted number of applicants. Two simple problems that naturally arise in this many-to-many matching context are those of assigning each applicant to all the courses she desires (hereafter, we refer to applicants as females) and of assigning each applicant to at least one course. Both cases reduce to well-known combinatorial optimization problems, namely the maximum flow problem and the maximum cardinality bipartite matching problem respectively, see e.g. [1].

In this paper, we allow several additional rules to apply. For example, each applicant, if accepted, may have to pay some fee as a contribution to cover the running costs of the course and she may have a budget that she is able to allocate to these courses that she cannot exceed. Alternatively, the courses may be of several different types, e.g., sports, language courses, summer schools etc. and the applicant may wish to attend at most one course of each type. The above rules are clearly pertinent to a number of real-life settings.

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Further, we shall also assume, like it is often the case in real life, that the applicants do not desire equally all the courses they apply for; rather they have certain preferences over them. The problem that arises when taking these preferences into account will be called the *Course Allocation problem* (CAP). In the CAP setting, various optimization criteria for the obtained assignments can be formulated. Here, we shall focus on *Pareto optimality*.

Pareto optimality, sometimes called *Pareto efficiency*, is a well established notion in economic science. It is the primary welfare goal in many real matching markets, especially educational markets assigning pupils to schools (see [2,3] for assigning students to public schools in several US school districts, [4] for college admission in Turkey) or students to campus housing [5,6]. A detailed account of recent developments regarding Pareto optimality in the context of matching problems under preferences has appeared in [7].

The special case of one-to-one CAP is often called the House Allocation problem, as it arises in the context of assigning tenants to houses [8,9]. Computational aspects of the House Allocation problem were studied in [10]. The authors gave necessary and sufficient conditions for a matching to be Pareto optimal and showed that these conditions can be checked in polynomial time. They also established that any Pareto optimal matching (POM) can be obtained by the well-known *serial dictatorship mechanism* [8] and proposed an efficient algorithm to find a POM of maximum cardinality. Analogous results have been established in [11] for the *many-to-one* (*capacitated*) House Allocation problem, i.e., the variant where each house can accommodate more than one tenant.

Regarding intractability results, it has been proved that finding a Pareto optimal matching of minimum cardinality is NP-complete even in the one-to-one case [10]. A related recent paper [12] deals with the computational complexity of serial dictatorship. The authors prove that in this mechanism, the problem of deciding whether there exists an order of proposals such that a given agent receives a given object is NP-complete, while the problem asking whether in each order of proposals a given agent receives a given object can be decided in polynomial time.

The remainder of this paper is organized as follows. In Section 2, we introduce our model in detail. Then, in Section 3, we completely characterize Pareto optimal matchings in the many-to-many setting and show that deciding whether a given matching is Pareto optimal can be accomplished in polynomial time. Consequently, our work concludes the research of [10,11] on these issues for two-sided markets as it treats the general case. In Section 4, we generalize the serial dictatorship mechanism, thus providing a procedure that can be used to obtain any many-to-many POM. This result is important also because, unlike in the one-to-one case (see [10]), serial dictatorship alone cannot guarantee that all Pareto optimal matchings will be generated. Section 5 is devoted to some structural questions. It is known [10,7] that the minimum cardinality POM problem is NP-complete, even when considering the simplified one-to-one case with the preference list of each applicant containing at most two entries, and we prove that the maximum cardinality POM problem is also NP-complete, although it is polynomially solvable in the (capacitated) House Allocation case [10,7,11].

2. Definitions

An instance of the Course Allocation problem involves a set A of n applicants and a set C of m courses. Each course $c \in C$ has a positive integral quota $q(c)$. A subset $A' \subseteq A$ of applicants is *feasible* for a course c if $|A'| \leq q(c)$. Each applicant a has a preference list $P(a)$, a strictly ordered list of a subset of courses. These courses are *acceptable* for a and we shall write $c \succ_a c'$ if applicant a prefers course c to course c'.

However, in general, we do not require that *any* subset of the acceptable courses is feasible for an applicant. By contrast, we assume that with each applicant $a \in A$ a family F_a of feasible sets is associated. We say that sets of courses that belong to F_a are *feasible* for a, while all other sets are *infeasible*. Below we list some possible examples coming from real world applications.

- **Simple** CAP. Each applicant a has an upper bound $u(a)$ for the number of courses she can attend. Then
$$F_a = \{S \subseteq P(a) : |S| \leq u(a)\}.$$
- **Price–budget** CAP. Consider the case in which each course c has a nonnegative price $p(c)$ and each applicant a has a budget $b(a)$. Let $p(S)$ denote the total price of all courses in the subset $S \subseteq C$, i.e. $p(S) = \sum_{c \in S} p(c)$. Here
$$F_a = \{S \subseteq P(a) : p(S) \leq b(a)\}.$$
- **Price-and-time** CAP. Each course c has a price $p(c)$ and some positive time-requirement $t(c)$. Applicant a has a budget $b(a)$ and also some time restriction $\tau(a)$. If we denote by $t(S)$ the total time needed to attend all the courses in $S \subseteq C$, i.e. $t(S) = \sum_{c \in S} t(c)$, then the feasible sets of applicant a are
$$F_a = \{S \subseteq P(a) : p(S) \leq b(a) \text{ and } t(S) \leq \tau(a)\}.$$
- **Partition** CAP. Let C be partitioned into disjoint classes $C_1, C_2, \ldots, C_r$ and applicant a has nonnegative *quotas* $q_1(a), \ldots, q_r(a)$ that denote the maximum number of courses from each class that applicant a is willing to attend. Now
$$F_a = \{S \subseteq P(a) : |S \cap C_i| \leq q_i(a) \text{ for each } i = 1, 2, \ldots, r\}.$$

We shall assume that for each applicant a, the family F_a is *downward closed*, i.e. if $S \subseteq T$ and $T \in F_a$, then $S \in F_a$, too. Note that the House Allocation problem is obtained if $q(c) = 1$ for each $c \in C$ and all the feasible sets are just singletons, containing the acceptable courses for each applicant.