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2-balanced flows and the inverse 1-median problem in the Chebyshev space

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ABSTRACT

In this paper, we consider the 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm. We give an optimality criterion for this problem which enables us to solve the following inverse location problem by a combinatorial algorithm in polynomial time: Given n points $P_1, \dots, P_n \in \mathbb{R}^d$ with non-negative weights w_i and a point P_0 the task is to find new non-negative weights $\tilde{w}_i$ such that P_0 is a 1-median with respect to the new weights and $\|w - \tilde{w}\|_1$ is minimized. In fact, this problem reduces to a 2-balanced flow problem for which an optimal solution can be obtained by solving a fractional b-matching problem.

1. Introduction

Location problems are an important field of operations research with many practical applications. The roots of location problems can be seen in an essay of Pierre de Fermat where he asked the question how a single facility should be placed in the plane such that the sum of the distances from a given set of three points to the new facility is minimized. Afterwards, this model was generalized by Weber [1] who considered n points with positive weights and the task is to find a location that minimizes the sum of the weighted distances to the given points. Nowadays, this problem is called the 1-median problem in the Euclidean plane and is also known as the Fermat–Weber problem. It is usually solved by an iterative algorithm which is based on an idea of Weiszfeld (see e.g., Drezner et al. [2] and the references therein).

Recently, a lot of other location problems have been investigated. The number of facilities that are allowed to be located was not fixed to one any more and different spaces were considered. Typically, one distinguishes between continuous location problems, discrete location problems and network location problems. In the first case facilities have to be located in some d-dimensional space like $\mathbb{R}^d$ whereas in a discrete location problem there is only a discrete set of potential locations. In network location problems the facilities can be placed on the vertices of a graph or in the interior of an edge. In addition, several different objective functions were introduced. The most common is the minimization of the sum of the weighted distances (median problems). However, center problems where one is interested in minimizing the largest weighted distance to a point are also well studied.

This paper focuses on the weighted 1-median problem in $\mathbb{R}^d$ where the distance of two points is measured by the Chebyshev-norm. This problem can be solved in linear time for $d = 2$ (see e.g., Hamacher [3]), because in this special case the Chebyshev-metric and the Manhattan-metric are related by a linear transformation. However, for $d \geq 3$ the same idea does not work any more, because the topology of these two norms is totally different. Hatzl and Karrenbauer [4] propose a first combinatorial algorithm for the d-dimensional case by reducing the problem to a min-cost-flow problem in a bipartite graph. Moreover, they prove that there is a 1-median, which is half-integral, provided that the given points have integral coordinates.

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The aim of this paper is to solve the inverse 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm. In an inverse optimization problem, a feasible solution is already given and the task is to modify some parameters of the problem at minimum cost such that the given solution is optimal with respect to the modified instance. This kind of problem was investigated for many combinatorial optimization problems, e.g., shortest path problem, maximum flow problem, spanning tree problem. For further results and solution methods on inverse optimization problems we refer to the survey paper of Heuberger [5]. In an inverse location problem the locations of the facilities are already given and one is allowed to change the edge lengths (if the problem is defined on a network) and/or the vertex weights. Depending on the model, there may also be some bounds on the parameters that are allowed to be modified. Moreover, there are different ways how to measure the total cost for changing the parameters.

Burkard et al. [6] considered the inverse p-median problem on graphs where it is allowed to change the vertex weights within some bounds. They prove that this problem can be solved in polynomial time if the asymmetric ℓ_1-norm is used as cost function. They also show that if $p = 1$ and the problem is defined on a tree the problem reduces to a continuous knapsack-problem which can be solved in linear time. Later, the same authors developed in [7] an $\mathcal{O}(n^2)$ algorithm if the underlying graph is a cycle and $p = 1$. The case where it is allowed to change the edge lengths of the graph was recently discussed by Baroughi et al. [8]. Cai et al. [9] prove that the inverse 1-center problem is strongly $\mathcal{NP}$-hard if the weights can be changed. This result is quite interesting, because the 1-center problem can be solved in polynomial time, but its inverse problem is hard. However, if one is allowed to change the edge lengths and the graph is a tree, the inverse problem is again solvable in polynomial time [10].

Inverse continuous location problems were less considered so far. Burkard et al. [11] prove that the inverse Fermat–Weber problem with variable weights and the ℓ_1-norm as cost function can be solved by a combinatorial algorithm in $\mathcal{O}(n \log n)$ if the prespecified point does not coincide with a given point. Another continuous inverse 1-median problem is discussed in [6]. Here, the authors focus on the Manhattan-metric and assume that the costs for changing the weights depend on the vertices.

In this paper, we solve the inverse 1-median problem with respect to the Chebyshev-norm. This problem was not investigated so far, because there was no optimality condition known. As a consequence it was difficult to find a combinatorial algorithm. Here we state the first combinatorial optimality criterion. Based on this result we develop an efficient algorithm for the inverse location problem.

This paper is organized as follows: In the next section the problems under consideration are defined. In Section 3, some preliminary results for the two-dimensional case are given which are helpful in order to deal with higher dimensions. Then, we state an optimality criterion for the considered 1-median problem and show that the corresponding inverse problem can be described as a 2-balanced flow problem. In Section 5, we discuss how the 2-balanced flow problem (and therefore also the inverse 1-median problem) can be transformed to a fractional b-matching problem which can be solved in polynomial time.

2. Problem formulation

In this section, we formally define the 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm and the corresponding inverse problem. Let us start with the classical location problem: Given n points $P_1, \ldots, P_n$ with $P_i = (x_1^i, \ldots, x_d^i) \in \mathbb{R}^d$ for $i = 1, \ldots, n$ and associated non-negative weights $w_i \geq 0$ the task is to find a point $P^* = (x_1^*, \ldots, x_d^*) \in \mathbb{R}^d$ such that

$$\sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty \geq \sum_{i=1}^{n} w_i \, \|P_i - P^*\|_\infty$$

holds for all $P \in \mathbb{R}^d$, where $\|P_i - P^*\|_\infty := \max\left(|x_1^i - x_1^*|, \ldots, |x_d^i - x_d^*|\right)$ is the Chebyshev-norm. Such a point P^* is called 1-median.

Note that the problem of finding the 1-median with respect to the Chebyshev-norm in the d-dimensional space can be written as a linear programming problem. To see this let us introduce new variables z_i which give the distance from P_i to the 1-median and rewrite the problem

$$\min_{P=(y_1,\ldots,y_d)\in\mathbb{R}^d} \sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty$$

as

$$\begin{aligned} \min \; & \sum_{i=1}^{n} w_i \, z_i \\ \text{s.t.} \; & z_i = \max\left(|x_1^i - y_1|, \ldots, |x_d^i - y_d|\right) \quad i = 1, \ldots, n \\ & y_j \in \mathbb{R}, \; z_i \in \mathbb{R} \quad j = 1, \ldots, d, \; i = 1, \ldots, n. \end{aligned} \tag{1}$$