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An algorithm for finding a maximum t-matching excluding complete partite subgraphs

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ABSTRACT

For an integer t and a fixed graph H, we consider the problem of finding a maximum t-matching not containing H as a subgraph, which we call the H-free t-matching problem. This problem is a generalization of the problem of finding a maximum 2-matching with no short cycles, which has been well-studied as a natural relaxation of the Hamiltonian circuit problem. When H is a complete graph K_{t+1} or a complete bipartite graph $K_{t,t}$, in 2010, Bérczi and Végh gave a polynomial-time algorithm for the H-free t-matching problem in simple graphs with maximum degree at most $t+1$. A main contribution of this paper is to extend this result to the case when H is a t-regular complete partite graph. We also show that the problem is NP-complete when H is a connected t-regular graph that is not complete partite. Since it is known that, for a connected t-regular graph H, the degree sequences of all H-free t-matchings in a graph form a jump system if and only if H is a complete partite graph, our results show that the polynomial-time solvability of the H-free t-matching problem is consistent with this condition.

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1. Introduction

For an undirected graph $G = (V, E)$ and an upper bound $b : V \to \mathbb{Z}_+$, an edge set $M \subseteq E$ is called a *simple b-matching* if the number of edges in M incident to v is at most $b(v)$ for each vertex $v \in V$. (We call such an edge subset just as a *b-matching* for remaining of the paper.) For a positive integer t, a b-matching with $b(v) = t$ for every vertex $v \in V$ is called a *t-matching*. In this paper, we deal with the problem of finding a maximum t-matching with some conditions in a given undirected graph.

When $t = 2$, the problem of finding a maximum 2-matching without short cycles has been studied as a natural relaxation of the Hamiltonian circuit problem. We say that a 2-matching is *$C_{\le k}$-free* if it contains no cycle of length k or less. The *$C_{\le k}$-free 2-matching problem* is to find a maximum $C_{\le k}$-free 2-matching. The case $k \le 2$ is exactly the classical simple 2-matching problem, which can be solved in polynomial time. Papadimitriou showed that the problem is NP-hard when $k \ge 5$ (see [1]). Hartvigsen provided an augmenting path algorithm for the $C_{\le 3}$-free 2-matching problem in his Ph.D. Thesis [2]. The computational complexity of the C_4-free 2-matching problem is still open, and several results are known in some graph classes. For the $C_{\le 4}$-free 2-matching problem in bipartite graphs, a min–max formula [3] and polynomial-time algorithms [4,5] are proposed. If each vertex of the input graph has degree at most three, a polynomial-time algorithm for finding a maximum 2-matching without cycles of length four and one for the $C_{\le 4}$-free 2-matching problem are given in [6] and [7], respectively. Note that 2-matchings containing no cycle of length four are closely related to the vertex-connectivity augmentation problem (see [6,8]).

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The weighted versions of those problems are to find a $C_{\leq k}$-free 2-matching that maximizes the total weight of its edges for a given weighted graph. This problem is NP-complete when $k \geq 4$ (see [6,9]), and it is left open when $k = 3$. The weighted $C_{\leq 4}$-free 2-matching problem in bipartite graphs is polynomially solvable if the weight function satisfies a certain condition called "vertex-induced on every square" [10,11], and the weighted $C_{\leq 3}$-free 2-matching problem in graphs with maximum degree at most three can also be solved in polynomial time [12,13].

The concept of $C_{\leq k}$-free 2-matchings can be extended to t-matchings in the following ways. The problem of finding a maximum t-matching not containing a $K_{t,t}$ as a subgraph, called the *$K_{t,t}$-free t-matching problem*, was first considered by Frank [9]. When $t = 2$ and an input graph is simple and bipartite, this problem is exactly the $C_{\leq 4}$-free 2-matching problem. Similarly, the notion of *K_{t+1}-free t-matchings*, which are t-matchings not containing a K_{t+1} as a subgraph, is a generalization of that of $C_{\leq 3}$-free 2-matchings. For the $K_{t,t}$-free t-matching problem in bipartite graphs, a min–max formula is given by Frank [9] and a combinatorial algorithm by Pap [14,5]. Bérczi and Végh [7] considered a common generalization of the $K_{t,t}$-free t-matching problem and the K_{t+1}-free t-matching problem, where the forbidden subgraph list can include both $K_{t,t}$'s and K_{t+1}'s. They gave a min–max formula and a combinatorial algorithm for this problem in graphs with maximum degree at most $t + 1$.

As a further generalization of these problems, for a fixed graph H, we consider the problem of finding a maximum t-matching not containing H as a subgraph, which we call the *H-free t-matching problem*. Motivated by the result of Bérczi and Végh [7], in this paper, we focus on the case when H is a connected t-regular graph and the input graph has maximum degree at most $t + 1$. Note that a graph is said to be *t-regular* if the degree of every vertex is t. Our main results are the following theorems, which draw a line between polynomially solvable cases and NP-hard cases.

Theorem 1. *Let t be a positive integer and H be a connected t-regular graph. If H is a complete partite graph, then the H-free t-matching problem in simple graphs with maximum degree at most $t + 1$ can be solved in polynomial time.*

Theorem 2. *Let t be a positive integer and H be a connected t-regular graph. If H is not a complete partite graph, then the H-free t-matching problem is NP-hard even if the input graph is simple and has maximum degree at most $t + 1$.*

Here, a graph $H = (V, E)$ is said to be a *complete partite graph* if there exists a partition $\{V_1, \dots, V_p\}$ of V such that $E = \{uv \mid u \in V_i, v \in V_j, i \neq j\}$ for some positive integer p. In other words, a complete partite graph is the complement of the disjoint union of the complete graphs. Since a K_{t+1} and a $K_{t,t}$ are t-regular complete partite graphs, Theorem 1 implies the polynomial-time solvability of the cases when $H = K_{t+1}$ and $H = K_{t,t}$.

We note that these theorems are also motivated by a relationship to jump systems. Jump systems are understood as a framework of efficiently solvable problems, and several relationships between jump systems and combinatorial optimization problems have been revealed [6,15]. Recently, it was shown in [16] that, for a connected t-regular graph H, the degree sequences of all H-free t-matchings in a graph form a jump system if and only if H is a complete partite graph. Thus, it is natural to consider the H-free t-matching problem for a complete partite graph H. The main results in the present paper, Theorems 1 and 2, show that the polynomial-time solvability of the H-free t-matching problem is consistent with this class of graphs H. It should be mentioned here that [16] is the first paper focusing on the class of complete partite graphs in the context of the H-free t-matching problem.

This paper is organized as follows. Before we move onto the proof of our main results, the preliminary for this paper is presented in Section 2. In Section 3, we show a slight generalization of Theorem 1, and give a polynomial-time algorithm for it. To show the correctness of our proof, we need a key theorem (Theorem 5), whose proof is given in Section 4. A proof for Theorem 2 is given in Section 5.

2. Preliminaries

Let $G = (V, E)$ be an undirected graph (or simply a graph) with vertex set V and edge set E, and n and m denote the number of vertices and the number of edges, respectively. For a vertex $v \in V$, the set of vertices adjacent to v is denoted by $N(v)$. The *degree* of a vertex $v \in V$ in G, denoted by $d_G(v)$, is the number of edges incident with v. For a vertex $v \in V$ and an edge set $F \subseteq E$, $d_F(v)$ is the number of edges in F incident with v. Note that if a self-loop e is incident with v, e is counted twice. Recall that, for a vector $b : V \to \mathbb{Z}_+$, an edge set $M \subseteq E$ is said to be a *b-matching* if $d_M(v) \leq b(v)$ for every $v \in V$. In particular, for a positive integer t, a b-matching with $b(v) = t$ for every vertex $v \in V$ is called a *t-matching*. Note that these are often called a *simple* b-matching and a *simple* t-matching in the literature. For a subgraph H of G, the vertex set and edge set of H are denoted by $V(H)$ and $E(H)$, respectively.

For a positive integer p, we say that a graph $H = (V, E)$ is a *complete p-partite graph* if there exists a partition $\{V_1, \dots, V_p\}$ of V, where $V_i \neq \emptyset$ for each i, such that $E = \{uv \mid u \in V_i, v \in V_j, i \neq j\}$. Each V_i in this partition is called a *color class*. A graph is *complete partite* if it is a complete p-partite graph for some p. A graph is *t-regular* if the degree of every vertex is t. One can easily observe that, for positive integers t and p, a t-regular complete p-partite graph exists if and only if $q := t/(p - 1)$ is an integer, and such a graph contains q vertices in each color class. For example, a t-regular complete 2-partite graph is a complete bipartite graph $K_{t,t}$, and a t-regular complete $(t + 1)$-partite graph is a complete graph K_{t+1}.

For a set $\mathcal{K}$ of subgraphs of $G = (V, E)$, an edge set $F \subseteq E$ is said to be *$\mathcal{K}$-free* if the graph $G_F = (V, F)$ contains no member of $\mathcal{K}$ as a subgraph. If $\mathcal{K}$ consists of all subgraphs of G that are isomorphic to H for some graph H, then a $\mathcal{K}$-free edge set is

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