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Non-zero-sum reinsurance games subject to ambiguous correlations

Chi Seng Pun [a], Chi Chung Siu [b], Hoi Ying Wong [c,*]

[a] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore
[b] Finance Discipline Group, UTS Business School, The University of Technology, Sydney, NSW, Australia
[c] Department of Statistics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

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ABSTRACT

This paper studies the economic implications of ambiguous correlation in a non-zero-sum game between two insurers. We establish the general framework of Nash equilibrium for the coupled optimization problems. For the constant absolute risk aversion (CARA) insurers, we show that the equilibrium reinsurance strategies admit closed-form solutions. Our results indicate that the ambiguous correlation leads to an increase in the equilibrium demand of reinsurance protection for both insurers. Numerical studies examine the effect on the quality of the correlation estimations.

1. Introduction

The optimal reinsurance and investment (IR) problems under different stochastic environments have been extensively studied in the fields of insurance and control theory. Representative works include but are not limited to [1,3,10,12,13]. However, the aforementioned studies do not take into account the effect of interactions among the insurance companies. In fact, economical and sociological studies have pointed out that human beings or firms tend to compare themselves to their peers, and that such relative performance concerns have significant impacts on one's decision-making. For example, [4] shows that the concept of relative performance concerns is relevant to financial bubbles and excess volatility. [6] establishes the unique existence of the Nash equilibrium for the optimal investment problems subject to the relative performance concerns in a N-agent economy under the Brownian motion framework. Subsequently, [2] extends the tractability of the non-zero-sum game framework to the IR problems with two insurers under the mixed regime-switching framework. [14] introduces model uncertainty into the associated IR games in [2], but does not effectively address the sensitivity of the correlation to the equilibrium strategies of the insurers.

In this paper, we study the robust reinsurance games between two insurers. Our present work differs from [2] in two key aspects. First, we treat the correlation coefficient (ρ) between two insurers' surplus processes as an ambiguous parameter which could be stochastic, whereas the correlation coefficient is a constant in [2]. Secondly, we allow the insurers to be either cooperative or competitive to highlight the impact of the ambiguous correlation in the non-zero-sum games between two competitive as well as two cooperative insurers, whereas only the case of competitive insurers is considered in [2]. Each insurer has her own confidence interval for ρ, where the bounds could be different (different constraints sets), and she maximizes her expected utility of her relative terminal surplus with respect to that of her counterparty by choosing her proportional reinsurance protection under the worst-case scenario of ρ. We show that the associated reinsurance game with ambiguous correlation fits naturally into the two-dimensional G-Brownian motion framework that is first introduced in [11] and has been subsequently applied to other stochastic control problems, as shown in [5,7].

Using the dynamic programming principle, we provide the Nash equilibrium of the robust non-zero-sum stochastic differential reinsurance game as the solution of a system of coupled Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations, for general utility functions. More importantly, we show that the Nash equilibrium reinsurance strategies and value functions of the insurers

* Corresponding author.
E-mail addresses: cspun@ntu.edu.sg (C.S. Pun), chichung.siu@uts.edu.au (C.C. Siu), hywong@cuhk.edu.hk (H.Y. Wong).

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admit explicit solutions for the case of constant absolute risk aversion (CARA) insurers. Our results indicate that the ambiguity in correlation leads to an increase in the demand of the reinsurance protections for both insurers, whether they are cooperative or competitive. Furthermore, our welfare analysis shows that the nature of externalities in the game between competitive insurers is different from that in the game between two cooperative insurers. To the best of our knowledge, the ambiguous correlation risks in the non-zero-sum reinsurance game has not been studied in the existing literature.

The rest of this paper is organized as follows. Section 2 formulates optimization problems of our interest with the surplus processes defined using G-Brownian motions. In Section 3, we apply the dynamic programming principle to the optimization problems and provide the sufficient conditions that the Nash equilibrium for the coupled problems exists. We also provide an explicit solution for the case of exponential utilities. Section 4 provides the numerical examples for the case of the CARA insurers, together with economic interpretations. Section 5 concludes the paper and discusses the possible extensions.

2. Problem formulation

We formulate the non-zero-sum game problem between two insurers using two-dimensional G-Brownian motion, which is introduced in [11]. That is, the associated game problem is studied in a complete space generated by the corresponding G-expectation. However, to best motivate the necessity of the G-Brownian motion framework, we shall begin with the standard insurance models under a physical measure $\mathbb{P}$ generated by the standard Brownian motion, and point out the ill-defined components under the assumption of the ambiguous correlation. We then apply the G-framework to reformulate our original game problem such that the mentioned ill-defined components become well-defined under the G-framework.

2.1. The model

We begin with the model of the surplus process of each insurer. Following [2], we adopt the standard Cramér–Lundberg diffusion approximation to model the surplus process of the insurer $k \in \{1, 2\}$, denoted by $\{X_k(t)\}_{t\geq 0}$. See [9] for the treatise on diffusion approximation in insurance models. Specifically, $X_k(t)$ satisfies the stochastic differential equation (SDE)

$$dX_k(t) = (p_k - \lambda_k \mathbb{E}[\eta_k])dt + \sqrt{\lambda_k \mathbb{E}[\eta_k^2]}\, d\mathcal{W}_k(t), \tag{1}$$

where $p_k > 0$ is the premium rate, $\lambda_k > 0$ is the arrival rate of the claims, $\eta_k \neq 0$ is a random variable representing the size of the claims with $\mathbb{E}[\eta_k^2] < \infty$ and $\{\mathcal{W}_k(t)\}_{t\geq 0}$ is a standard $\mathbb{P}$-Brownian motion for $k = 1, 2$. The dependence between two insurers is reflected by the correlation between $\{\mathcal{W}_1(t)\}_{t\geq 0}$ and $\{\mathcal{W}_2(t)\}_{t\geq 0}$, i.e. $d\mathcal{W}_1(t)d\mathcal{W}_2(t) = \rho(t)dt$, but $\rho(t)$ is uncertain and possibly stochastic, in the sense that the insurer k only knows $\rho(t) \in [\underline{\rho}_k, \overline{\rho}_k]$. In what follows, we shall assume that the parameters of the model dynamics are constants and independent of time. We do so to explicitly capture the consequence of the ambiguous correlation on the Nash equilibrium in the associated non-zero-sum game. The extension to the time-varying parameters is rather immediate but yields no additional economic insights.

Suppose that there is a reinsurance company, then the insurer $k \in \{1, 2\}$ can manage her insurance risks through purchasing proportional reinsurance protection at the premium rate $\theta_k > p_k > 0$. Let $1 - q_k(t)$ be the reinsurance proportion of the insurer $k \in \{1, 2\}$ at time t. Then the reinsurance company will cover $(1 - q_k(t))100\%$ of the claims while the insurer k will cover the remaining. The reinsurance strategy of the insurer $k \in \{1, 2\}$ is characterized by $\{q_k(t)\}_{t\geq 0}$, which is a $\mathcal{F}_t$-progressively measurable process valued in $[0, 1]$ and $\mathcal{F}_t = \sigma(\{(\mathcal{W}_1(s), \mathcal{W}_2(s))\}_{s=0}^t)$. We denote $Q_k = \{q_k(t) \in \mathcal{F}_t | q_k(t) \in [0, 1]\}$ as the set of convex reinsurance strategies of insurer k. With reinsurance, the surplus process $\{X_k^{q_k}(t)\}_{t\geq 0}$ of the insurer $k \in \{1, 2\}$ becomes

$$\begin{aligned} dX_k^{q_k}(t) &= [p_k - \theta_k(1 - q_k(t)) - \lambda_k \mathbb{E}[\eta_k] q_k(t)]dt \\ &\quad + \sqrt{\lambda_k \mathbb{E}[\eta_k^2]}\, q_k(t) d\mathcal{W}_k(t), \\ &=: [\delta_k + \mu_k q_k(t)]dt + \sigma_k q_k(t) d\mathcal{W}_k(t), \end{aligned} \tag{2}$$

where $\delta_k = p_k - \theta_k < 0$ is the premium difference, $\mu_k = \theta_k - \lambda_k \mathbb{E}[\eta_k]$ is the relative safety loading and $\sigma_k = \sqrt{\lambda_k \mathbb{E}[\eta_k^2]}$ is the volatility of the claims process. We assume the initial reserve of the insurer $k \in \{1, 2\}$ is $X_k^{q_k}(0) = x_k > 0$.

2.2. Objectives of the insurers

Suppose that the insurer $k \in \{1, 2\}$ has a utility function, denoted by U_k, which is increasing and strictly concave function valued in $\mathbb{R}$ and satisfies Inada conditions:

$$\left.\frac{\partial U_k}{\partial x}\right|_{x\to -\infty} = +\infty, \qquad \left.\frac{\partial U_k}{\partial x}\right|_{x\to +\infty} = 0.$$

To incorporate the ambiguity of the correlation and the interaction between two insurers, we assume the objective of each insurer is to maximize the expected utility of a linear combination of both insurers' surpluses at terminal time $T > 0$ under the worst-case scenario of the correlation. Mathematically, we consider the following optimization problem for the insurer $k \in \{1, 2\}$:

$$\sup_{q_k \in Q_k} \inf_{\rho \in [\underline{\rho}_k, \overline{\rho}_k]} \mathbb{E}\left[U_k(X_k^{q_k}(T) - \kappa_k X_m^{q_m}(T))\right] \tag{3}$$

for $m \neq k \in \{1, 2\}$, where $\kappa_k \in [-1, 1]$ reflects the level of relative performance concern of the insurer k. Indeed, when $\kappa_k = 0$, for $k = 1, 2$, we return to the single-agent optimal reinsurance problem, in which ambiguous correlation would play no role in the insurer's optimization problem. In light of this, we shall hereafter assume that $\kappa_k \neq 0$. When $\kappa_k \in (0, 1]$ (resp. $\kappa_k \in [-1, 0)$), for $k = 1, 2$, insurer k treats insurer m, for $k \neq m \in \{1, 2\}$ as competitor (resp. cooperator), as her optimization problem in (3) would indicate that she would optimally purchase reinsurance protection to maximize the difference between her terminal surplus against (resp. the sum of her terminal surplus with) that her competitor under the correlation estimate that yields the worse expected payoff. Although the Nash equilibrium in Section 3 also includes the case when insurer k is competitive ($\kappa_k > 0$) and insurer m is cooperative ($\kappa_m < 0$), for $k \neq m \in \{1, 2\}$, we choose not to study this case for there is no clear economic rationale on the establishment of a game between one competitive and one cooperative insurers. See [2,6,14] for the optimization under the relative performance concerns when $\kappa_k \in [0, 1]$; and [7] for the maximin formulation of the robust portfolio optimization with ambiguous correlation.

Major technical hurdle arising from our problem formulation is that the underlying measure of the expectation and the admissible set of the reinsurance strategies in (3) are not clear. More