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Folk solution for *simple* minimum cost spanning tree problems

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ABSTRACT

A *minimum cost spanning tree* problem analyzes how to efficiently connect a group of individuals to a source. Once the efficient tree is obtained, the addressed question is how to allocate the total cost among the involved agents. One prominent solution in allocating this minimum cost is the so-called *Folk* solution. Unfortunately, in general, the *Folk* solution is not easy to compute. We identify a class of *mcst* problems in which the *Folk* solution is obtained in an easy way. This class includes *elementary cost mcst* problems.

1. Introduction

We consider a situation in which some individuals, located at different places, want to be connected to a source in order to obtain a good or service. Each link joining two individuals, or any individual to the source, has a specific fixed cost. Moreover, individuals do not mind being connected directly to the source, or indirectly through other individuals. There are several methods to obtain a way of connecting agents to the source so that *the total cost of the selected network is minimum* (Prim's algorithm [1], for instance). This situation is known as the *minimum cost spanning tree problem* (hereafter *mcst* problem) and it is used to analyze different real-life issues, from telephone and cable TV to water supply networks.

An important question is how this minimum cost should be allocated among the individuals. One prominent solution to solve the allocation of this cost is the so-called *Folk* solution. To compute this solution, first we need to calculate the irreducible costs and, in order to do that, we have to compare all paths connecting any two nodes (individuals) and solve a min − max problem. Then, we have to compute the Shapley value of the cooperative game defined throughout the irreducible costs, or to apply the closed-form obtained in Bogomolnaia and Moulin [5].

We define a class of *mcst* problems (that we call *simple mcst* problems) in which the *Folk* solution only depends on the cost of each individual to the source and the cost to the *nearest partner*, that is, the minimum connection cost of this individual. We obtain a closed-form (easy to obtain) of the *Folk* solution that does not need to compute the irreducible costs. Finally, we extend the class of *mcst* problems where this procedure can be applied: *simple-decomposable mcst* problems, a class that includes elementary cost matrix problems.

2. Definitions

2.1. Minimum cost spanning tree

A *minimum cost spanning tree* problem involves a finite set of *agents*, $N = \{1, 2, \dots, n\}$, who need to be connected to a *source* ω. The agents are connected by edges and for $i \neq j$, $c_{ij} \in \mathbb{R}_+$ represents the cost of the edge $e_{ij} = (i, j)$ connecting agents $i, j \in N$. We denote by c_{ii} the cost of connecting directly agent i to the source, for all $i \in N$. Let $\mathbf{C} = [c_{ij}]_{n \times n}$ be the $n \times n$ symmetric cost matrix. The *mcst* problem is represented by the pair $(N, \mathbf{C})$.

A *spanning tree* over $(N, \mathbf{C})$ is an undirected graph p with no cycles that connects all elements of $N \cup \{\omega\}$. We can identify a spanning tree with a map $p : N \to N \cup \{\omega\}$ so that $j = p(i)$ is the agent (or the source) whom i connects. This map p defines the edges $e^p_{ij} = (i, p(i))$ in the tree. In a spanning tree each agent is (directly or indirectly) connected to the source ω; that is, for all $i \in N$ there is some $t \in \mathbb{N}$ such that $p^t(i) = \omega$. Moreover, given a spanning tree p, there is a unique path from any i to the source for all $i \in N$, given by the edges $(i, p(i)), (p(i), p^2(i)), \dots, (p^{t-1}(i), p^t(i) = \omega)$. The cost of building the spanning tree p is the total cost of the edges in this

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tree; that is, $C_p = \sum_{i=1}^{n} c_{ip(i)}$. Prim [1] provides an algorithm which solves the problem of connecting all agents to the source such that *the total cost of the network is minimum*. The achieved solution, the *minimum cost spanning tree*, may not be unique. Denote by m a tree with minimum cost and by C_m its cost. That is, for all spanning trees p

$$C_m = \sum_{i=1}^{n} c_{im(i)} \leqslant C_p = \sum_{i=1}^{n} c_{ip(i)}.$$

Once a minimum cost spanning tree is constructed, the important issue is how to allocate the associated cost C_m among the agents. A *cost sharing rule* for *mcst* problems is a function that proposes for any *mcst* problem $(N, \mathbf{C})$ an allocation $\alpha(N, \mathbf{C}) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, such that $\sum_{i=1}^{n} \alpha_i = C_m$.

Given a subset $S \subseteq N$, we will denote by $C_m(S)$ the minimum cost of the *mcst* sub-problem $(S, \mathbf{C}|_S)$. Let us denote by C_ω the cost of the tree in which every individual joins directly the source, $C_\omega = \sum_{i\in N} c_{ii}$. And, for any individual $i \in N$, c_{i_*} represents the minimum connection cost of such an individual (interpreted as the cost to his *nearest partner*), $c_{i_*} = \min_{j\in N} c_{ij}$. Note that the nearest partner can be the source ω, in which case $c_{i_*} = c_{ii}$.

2.2. The **Folk** solution

Many solutions have been defined in the *mcst* literature (see, for instance, Bergantiños and Vidal-Puga [4] for definitions and a comparative analysis). We will focus on the so-called *Folk* solution proposed independently by Feltkamp et al. [2] and Bergantiños and Vidal-Puga [3]. We will denote this solution by $F(N, \mathbf{C})$. It can be obtained as the Shapley value of the stand-alone game associated with the irreducible cost matrix defined by:

$$c^*_{ij} = \min_{P_{ij}} \max_{e\in P_{ij}} \{c(e)\}$$

where P_{ij} are paths from i to j, $e \in P_{ij}$ is an edge in this path, and $c(e)$ is the cost of this edge. It must be noticed that, with the notation we are using, P_{ii} denotes a path from individual i to the source ω.

Bogomolnaia and Moulin [5] provide a closed-form expression of the *Folk* solution: for individual i consider the irreducible costs of connecting this individual to other $n-1$ agents, c^*_{ik}, rearranged in increasing order. Then, the *Folk* solution is

$$F_i(N, \mathbf{C}) = \frac{c^*_{ii}}{n} + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \min\{c^*_{ik}, c^*_{ii}\}. \quad (1)$$

2.3. **Simple** *mcst* **problems**

Definition 1 (*Elementary Cost Problem*)**.** An *mcst* problem $(N, \mathbf{C})$ is said to be an **elementary cost** ***mcst*** **problem** if for all $i, j \in N$, $c_{ij} \in \{c_1, c_2\}$, with $c_1 < c_2$. We will denote an elementary cost *mcst* problem by $(N, \mathbf{C^e})$ and $\mathcal{E}$ will represent the class of elementary cost problems.

Remark 1. Usually, elementary cost matrices are defined such that $c_1 = 0$ and $c_2 = 1$. The general case $c_1 < c_2$, low and high cost, is also known as 2-*mcst* problems (Estévez-Fernández and Reijnierse [6]).

Definition 2 (*Autonomous Component*)**.** Given an *mcst* problem $(N, \mathbf{C})$, with minimum connecting cost C_m, a subset $S \subseteq N$ is said to be:

- **autonomous** if $C_m = C_m(S) + C_m(N \setminus S)$;
- **an autonomous component** if it is autonomous and has no proper subset that is also autonomous; if $T \subseteq S$, $T \neq S$, then T is not autonomous.

Remark 2. Obviously N is always autonomous. If it is an autonomous component, it is the unique autonomous component in the *mcst* problem.

Definition 3 (*Simple mcst Problem*)**.** Given an *mcst* problem $(N, \mathbf{C})$, it is said to be **simple** if it is an elementary cost problem and the set of all individuals N is an autonomous component. We will denote a simple *mcst* problem by $(N, \mathbf{C^s})$ and $\mathcal{S}$ will represent the class of simple problems.

3. Results

The following results show that in *simple mcst problems* it is possible to obtain the *Folk* solution only taking into account, for each individual $i \in N$, the cost of connecting this individual to the source, c_{ii}, and the cost to connect with his nearest partner c_{i_*}.

Lemma 1. *Given a* ***simple*** *mcst problem* $(N, \mathbf{C^s}) \in \mathcal{S}$,

(a) *There is at most one individual* $i \in N$ *such that* $c_{ii} = c_1$.
(b) *For all* $i \in N$, $c_{i_*} = c_1$.
(c) *For all* $i, j \in N$, $i \neq j$, $c^*_{ij} = c_1$.

Proof. (a) Let us consider a simple *mcst* problem $(N, \mathbf{C^s})$. First observe that if two different individuals $i \neq j$ fulfill $c_{ii} = c_{jj} = c_1$ then individuals i and j can connect directly to the source, in an independent way, at the same cost and N is not an autonomous component, a contradiction.

To prove (b), observe that $c_{i_*} = c_2$ implies $S = \{i\}$ is autonomous, a contradiction. Therefore, $c_{i_*} = c_1$ for all $i \in N$. Finally note that (a) and (b) imply that $c_{im(i)} = c_1$, for each $i \in N$, such that $m(i) \neq \omega$. Then, for $i, j \in N$, $i \neq j$, it is possible to find a path P_{ij} such that $c(e) = c_1$ for all $e \in P_{ij}$ (for instance, throughout the individual k, such that $m(k) = \omega$). So $c^*_{ij} = c_1$, for all $i \neq j$. ■

Theorem 1. *Given a* ***simple*** *mcst problem* $(N, \mathbf{C^s}) \in \mathcal{S}$

(a) *If there exists* $i \in N$ *such that* $c_{ii} = c_1$ *then* $F_j(N, \mathbf{C^s}) = c_1$ *for all* $j \in N$.
(b) *If* $c_{ii} = c_2$ $\forall i \in N$ *then* $F_j(N, \mathbf{C^s}) = \frac{C_m}{n}$ *for all* $j \in N$.

Proof. (a) In this case $c^*_{ii} = c_1$ and from Lemma 1 $c^*_{ij} = c_1$ for all $i, j \in N$, so the result is obtained by computing the *Folk* solution through Eq. (1).

(b) If there is no individual who can connect the source with low cost, by Lemma 1 we know that for all $i \neq j \in N$, $c^*_{ij} = c_1$, and $c^*_{ii} = c_2$. Since coefficients in Eq. (1) coincide for all the individuals, the *Folk* solution gives an equal sharing of the cost. Notice that, as in this case Lemma 1 implies $C_m = c_2 + (n-1)c_1$, then $F_j(N, \mathbf{C^s}) = \frac{C_m}{n} = c_1 + \frac{c_2 - c_1}{n}$ for all $j \in N$. ■

Now, we extend the class of *mcst* problems in which the result in Theorem 1 can be obtained by allowing problems with several autonomous components.

Definition 4. An *mcst* problem $(N, \mathbf{C})$ is **simple-decomposable** if it is possible to split N

$$N = N_1 \cup N_2 \cup \cdots \cup N_r, \qquad N_t \cap N_{t'} = \emptyset, \quad \text{for } t \neq t'$$

such that

$$C_m(N, \mathbf{C}) = \sum_{t=1}^{r} C_m\left(N_t, \mathbf{C}|_{N_t}\right)$$