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Geometric proofs for convex hull defining formulations

Mark Zuckerberg *

Department of Mathematics and Statistics, Melbourne University, Victoria 3010, Australia
BHP Billiton, Technology, 171 Collins Street, Melbourne, Victoria 3000, Australia

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ABSTRACT

A conjecture appeared recently in Cacchiani et al. (2013) that a proposed LP relaxation of a certain integer programming problem defines the convex hull of its integer points. We review a little known technique described in Zuckerberg (2004) that can be used to construct geometric proofs that an LP relaxation is convex hull defining. In line with this technique, we show that their conjecture is correct.

1. Introduction

The connection between probability measures and integer programming has antecedents in [6] (in the "Remark" on page 186), and earlier (see Part 1 in [4]), with later development in [2] (beginning in Section 2.1) and in [8] (Chapters 3 and 4), where this connection is shown to generalize and contextualize the lifting methods of [1,7,6,5]. In the course of the analysis in [8], a method is described by which an LP relaxation of an integer programming formulation can be proven to define the convex hull of its integer points if it can be shown that for each point in R^n that is in the linear relaxation, it is possible to draw n sets in an arbitrary measure space that have properties that match the logical properties of the integer feasible set in a certain way. The method is described as "geometric", or as in [8], as the "proof by picture method", as one can potentially demonstrate that a formulation is convex hull defining by physically drawing sets in the real line or plane with the requisite properties. In this work we will provide a compact description and rigorous justification of the method, and we will use it to resolve a conjecture that appeared in [3].

2. Binary integer programming and probability measures

First we review some basic definitions and facts.

Definition 1. 1. Given a set U and a family $\mathcal{L} \neq \emptyset$ of subsets of U, the pair $\mathcal{Q} = (U, \mathcal{L})$ is said to be an "algebra on U" if $\mathcal{L}$ is closed under unions, intersections and complementations, where complementation is defined as $h^c \doteq U \setminus h,\ \forall h \in \mathcal{L}$. Note that for any algebra $\mathcal{Q} = (U, \mathcal{L}),\ U \in \mathcal{L}$. Given an algebra $\mathcal{Q} = (U, \mathcal{L})$, we will refer to a set h as being "in algebra $\mathcal{Q}$" if $h \in \mathcal{L}$.
2. A nonempty set a in an algebra $\mathcal{Q} = (U, \mathcal{L})$ is called an "atom" of $\mathcal{Q}$ if for every $h \in \mathcal{L}$, either $a \cap h = a$ or $a \cap h = \emptyset$, i.e. a is indivisible.
3. An algebra $\mathcal{Q} = (U, \mathcal{L})$ is "generated by" sets $\{S_i\} \subseteq \mathcal{L}$ if all sets in $\mathcal{L}$ can be written as finitely many unions, intersections and complementations of the sets in $\{S_i\}$.
4. If $\mathcal{L}$ is the collection of all subsets of a set U, then $\mathcal{Q} = (U, \mathcal{L})$ is said to be the *subset algebra* of U.
5. Denote the subset algebra of $\{0, 1\}^n$ as $\mathcal{A}$. Note that the atoms of $\mathcal{A}$ are the points of $\{0, 1\}^n$ (i.e. the singleton sets).
6. For each $i = 1, \ldots, n$, define $A_i = \{y \in \{0, 1\}^n : y_i = 1\}$.
7. Let $\mathcal{Q} = (U, \mathcal{L})$ be an algebra. A set function χ mapping $\mathcal{L}$ to $[0, 1]$ is a "probability measure on $\mathcal{Q}$" if $\chi(U) = 1$, and if for each disjoint pair $p, q \in \mathcal{L}$, $\chi(p \cup q) = \chi(p) + \chi(q)$.

Remark 2. The atoms of the algebra $\mathcal{Q} = (U, \mathcal{L})$ generated by $\{S_1, \ldots, S_n\} \subseteq \mathcal{L}$ are the nonempty sets of the form $\bigcap_{i \in H} S_i \cap \bigcap_{i \in H^c} S_i^c$, where $H \subseteq \{1, \ldots, n\}$. (This follows from an easy adaptation of Lemma 5, as we will note in its proof.) It follows then that every set in $\mathcal{L}$ is a finite union of these sets, as they are indivisible and partition U (as every point in U has a unique membership profile in the $\{S_i\}$). Abusing convention, we will

* Correspondence to: Department of Mathematics and Statistics, Melbourne University, Victoria 3010, Australia.
E-mail addresses: zum@unimelb.edu.au, mark.zuckerberg@bhpbilliton.com.

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occasionally refer to sets of this form as atoms even if they are empty. Note also that for infinite algebras, more care is usually taken around the definition of probability measures, but this need not concern us here, as our focus will be on finite algebras.

Lemma 3. *The algebra $\mathcal{A}$ is generated by $\{A_i,\ i = 1, \dots, n\}$.*

Proof. It is sufficient to show that the atoms of $\mathcal{A}$, which are the points of $\{0, 1\}^n$, can be written as set theoretic expressions of $\{A_i\}$, as all sets in $\mathcal{A}$ are obviously unions of these. For any $y \in \{0, 1\}^n$, let $H \subseteq \{1, \dots, n\}$ be the indices h for which $y_h = 1$. Then $\{y\} = \bigcap_{h\in H} A_h \cap \bigcap_{h\in H^c} A_h^c$. ■

The following lemma, from [8], establishes the connection between binary integer programming and probability measures. It shows that the convex hull of a set $\mathcal{F} \subseteq \{0, 1\}^n$ is the set of points $x \in R^n$ that can be lifted to a probability measure χ on $\mathcal{A}$ satisfying $\chi(\mathcal{F}) = 1$.

Lemma 4. *Let $\mathcal{F} \subseteq \{0, 1\}^n$. The point $x \in [0, 1]^n$ belongs to the convex hull of $\mathcal{F}$ iff there exists a probability measure χ on $\mathcal{A}$ such that $\chi(\mathcal{F}) = 1$ and*

$$x_i = \chi(A_i), \quad \forall i = 1, \dots, n. \tag{1}$$

Proof. Let $\{y^1, \dots, y^r\}$ be an enumeration of all points in $\mathcal{F}$, and assume $x \in Conv(\mathcal{F})$. Then there exist nonnegative multipliers $\{\lambda_1, \dots, \lambda_r\}$, with $\sum_{i=1}^r \lambda_i = 1$, such that $x = \sum_{i=1}^r \lambda_i y^i$. Noting that the points of $\{0, 1\}^n$ are the atoms of the finite algebra $\mathcal{A}$, it is sufficient to define χ on these points to uniquely define the measure χ by additivity (i.e. for any $G \subseteq \{0, 1\}^n$, $\chi(G) = \sum_{y\in G} \chi(y)$, where we have written $\chi(y)$ instead of $\chi(\{y\})$ to reduce clutter). Assign $\chi(y^i) = \lambda_i$ for all i and $\chi(y) = 0$ for all $y \in \mathcal{F}^c$. Since $\sum_{i=1}^r \lambda_i = 1$, it follows that χ is a probability measure, and $\chi(\mathcal{F}) = \sum_{y\in\mathcal{F}} \chi(y) = \sum_{i=1}^r \lambda_i = 1$. Moreover,

$$\chi(A_i) = \sum_{y\in\{0,1\}^n : y_i=1} \chi(y) = \sum_{y\in\mathcal{F} : y_i=1} \chi(y) = \sum_{j\in\{1,\dots,r\} : y_i^j=1} \lambda_j$$
$$= \sum_{j=1}^r \lambda_j y_i^j = x_i. \tag{2}$$

Conversely, if χ is a probability measure on $\mathcal{A}$ with $\chi(\mathcal{F}) = 1$, and $\chi(A_i) = x_i,\ i = 1, \dots, n$, then for each $j = 1, \dots, r$, let $\lambda_j = \chi(y^j)$, so that $\lambda \geq 0$ and $\sum_{j=1}^r \lambda_j = \chi(\mathcal{F}) = 1$, and observe that for each $i = 1, \dots, n$,

$$x_i = \chi(A_i) = \sum_{y\in\{0,1\}^n : y_i=1} \chi(y) = \sum_{y\in\mathcal{F} : y_i=1} \chi(y) = \sum_{j\in\{1,\dots,r\} : y_i^j=1} \lambda_j$$
$$= \sum_{j=1}^r \lambda_j y_i^j \tag{3}$$

which establishes that $x \in Conv(\mathcal{F})$. ■

The main result in this section, which will justify the "proof by picture" method, states that the convex hull of a set can be characterized not only by probability measures on $\mathcal{A}$, but also by probability measures on arbitrary algebras. First, however, we need a preliminary result.

In what follows, a "logical expression" is an expression comprised of symbols representing Boolean variables, conjunctions, disjunctions and negations, and a "set theoretic expression" is one comprised of symbols representing sets, unions, intersections and complementations. We may occasionally conflate an expression with its value, if the meaning is clear.

The following lemma concerns the relationship between logical expressions of Boolean variables and set theoretic expressions, and between truth assignments of Boolean variables and atomic sets. In summary, it states that the points in a nonempty atom belong to a set if and only if the truth assignment associated with that atom satisfies the logical statement associated with that set.

Lemma 5. *Let U be a set with subsets $\{S_1, \dots, S_n\}$, and let $\mathcal{Q} = (U, \mathcal{L})$ be the algebra generated by $\{S_1, \dots, S_n\}$. Let $E(\{S_i\})$ be a set theoretic expression of finitely many unions, intersections and complementations of sets from $\{S_1, \dots, S_n\}$. Define $\{B_1, \dots, B_n\}$ to be n Boolean variables, and let $L(E(\{B_i\}))$ be the logical expression obtained by replacing each union in $E(\{S_i\})$ with logical "OR", each intersection with logical "AND", each complementation with logical "NOT" and each S_i with the Boolean variable B_i. Let $H \subset \{1, \dots, n\}$, and consider the atom $a = \bigcap_{i\in H} S_i \cap \bigcap_{i\in H^c} S_i^c$, and assume $a \neq \emptyset$. Then a is a subset of the set defined by $E(\{S_i\})$ if and only if the logical statement $L(E(\{B_i\}))$ is true for the instantiation of variables (i.e. the "truth assignment") that assigns B_i to true for each $i \in H$ and false for each $i \in H^c$.*

Proof. The proof will be by induction. Note first that every $H \subseteq \{1, \dots, n\}$ defines both an atom $\bigcap_{i\in H} S_i \cap \bigcap_{i\in H^c} S_i^c$, as well as a truth assignment B_i = "true", $i \in H$, B_i = "false", $i \in H^c$. Consider the case for which $E(\{S_i\})$ is the single symbol S_j for some j. Clearly, a nonempty atom a defined by $H \subseteq \{1, \dots, n\}$ satisfies $a \subseteq E(\{S_i\})$ (taken to mean that a is subset to the set defined by expression $E(\{S_i\})$) iff $j \in H$, and similarly $L(E(\{B_i\})) = B_j$ is true for exactly those truth assignments with $j \in H$. Now assume that the lemma holds for some pair of expressions $E_1(\{S_i\})$ and $E_2(\{S_i\})$, and consider $E(\{S_i\})$ written as $E_1(\{S_i\}) \cup E_2(\{S_i\})$. For any $H \subseteq \{1, \dots, n\}$, the atom a defined by H satisfies $a \subseteq E(\{S_i\})$ iff $a \subseteq E_1(\{S_i\})$ or $a \subseteq E_2(\{S_i\})$, iff the truth assignment defined by H makes $L(E_1(\{B_i\}))$ or $L(E_2(\{B_i\}))$ true, iff it makes $L(E(\{B_i\}))$ true. The argument is similar for intersections and complementations. (Note that the argument in this proof can be applied to each point in a individually as well, which constitutes a proof that these sets really are indivisible.) ■

Corollary 6. *Let $E_1(\{A_i\})$ and $E_2(\{A_i\})$ be two finitely long set theoretic expressions of sets from $\{A_1, \dots, A_n\}$ that define the same set. Then for any set U with subsets $\{S_1, \dots, S_n\}$ generating an algebra $\mathcal{Q}$, $E_1(\{S_i\})$ and $E_2(\{S_i\})$ define the same set as well.*

Proof. Since every set in $\mathcal{Q}$ is a finite union of atoms, to establish that $E_1(\{S_i\}) = E_2(\{S_i\})$ we need only show, by Lemma 5, that for every nonempty atom defined by $H \subseteq \{1, \dots, n\}$, the associated assignment of truth values to n Boolean variables $B_1, \dots, B_n$ makes $L(E_1(\{B_i\}))$ true iff it makes $L(E_2(\{B_i\}))$ true. Observe that for every $H \subseteq \{1, \dots, n\}$, the corresponding atom $\bigcap_{i\in H} A_i \cap \bigcap_{i\in H^c} A_i^c$ is comprised of the point with 1's in its indices $i \in H$ and 0's in the other indices, and so all atoms of $\mathcal{A}$ are nonempty. The condition therefore to establish $E_1(\{A_i\}) = E_2(\{A_i\})$ is at least as strong as the condition required to establish $E_1(\{S_i\}) = E_2(\{S_i\})$. ■

Now we can prove the main theorem of this section.

Theorem 7. *Let $\mathcal{F} \subseteq \{0, 1\}^n$ and let $F(\{A_i\})$ be a set theoretic expression of finitely many unions, intersections and complementations of sets from $\{A_1, \dots, A_n\}$ that equals $\mathcal{F}$. Let $\mathcal{Q} = (U, \mathcal{L})$ be any algebra, and let ξ be any probability measure on $\mathcal{Q}$. Then $x \in [0, 1]^n$ belongs to $Conv(\mathcal{F})$ if we can find sets $S_i \in \mathcal{L},\ i = 1, \dots, n$, with $x_i = \xi(S_i)$ for each i, and such that $\xi(F(\{S_i\})) = 1$.*

Proof. The subalgebra $\mathcal{Q}^S$ of $\mathcal{Q}$ on U generated by $\{S_i\}$ is also an algebra, and ξ remains a probability measure on $\mathcal{Q}^S$. We claim that if we define χ by $\chi(E(\{A_i\})) = \xi(E(\{S_i\}))$ for every set theoretic expression E, then χ is a probability measure on $\mathcal{A}$. To prove the claim we must first show that χ is a valid set function, i.e. it assigns a single value to every set in $\mathcal{A}$. It is easy to see that χ assigns