Operations Research Letters 44 (2016) 649–654

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Complexity of the Unconstrained Traveling Tournament Problem

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ARTICLE INFO

Article history:
Received 3 August 2015
Received in revised form
15 July 2016
Accepted 16 July 2016
Available online 22 July 2016

Keywords:
Scheduling
Traveling Tournament Problem
NP-hard

ABSTRACT

The Traveling Tournament problem is a problem of scheduling round robin leagues which minimizes the total travel distance maintaining some constraints on consecutive home and away matches. The problem was proven NP-hard when the upper bound on any consecutive home or away stint is 3. In this paper, we prove that even without the constraints on the consecutive home or away matches, the problem remains NP-Hard.

1. Introduction

The *Traveling Tournament Problem* (TTP) addresses the problem of minimizing the total travel distance in a double round robin league tournament schedule where each team can play at most U consecutive home and away matches [4]. Since its proposal in [4], TTP has been a notoriously difficult problem to solve. Numerous works using metaheuristic techniques [1,2,5] have been conducted to solve TTP in the past years. We refer the reader to [9,8] for detailed survey on these techniques. The complexity of TTP was also open for many years. Finally, in 2011, Thielen and Westphal proved that the Traveling Tournament problem with $U = 3$ is *NP*-hard [10].

Given the hardness of the constrained TTP, it is natural to explore a simplified problem. One natural is TTP without any constraint on the consecutive home and away matches. Formally, the Unconstrained Traveling Tournament Problem (UTTP) can be defined as follows:

- **[Input]**:
  - The number of teams n (n is even).
  - A distance matrix $D_{n\times n}$.
- **[Output]**: A double round robin league schedule of n teams such that the total distance traveled by all the teams is minimized.

Recently, Imahori, Matsui, and Miyashiro have proposed a 2.75-approximation algorithm for the Unconstrained TTP [7].

My result

In this paper, we prove that the Unconstrained Traveling Tournament Problem is *NP*-hard. Specifically, we show that if the teams are allowed to play any number of consecutive home and away matches, then there is a reduction from the (1, 2)- Traveling Salesman Problem; the problem of finding a TSP tour in a graph where the cost of every edge is either 1 or 2.

Theorem 1.1. *The Unconstrained Traveling Tournament problem is NP-hard.*

The rest of the paper is organized as follows. In the next section, we introduce the notations used in the paper and formally define the decision problems. In Section 3 we describe the constructions and the associated results needed for the reduction. Finally in Section 4 we describe the formal reduction from (1, 2)-TSP to UTTP.

2. Notation and preliminaries

Throughout the paper we follow the following notation. If S is a set $|S|$ denotes the cardinality of S. $[n]$ denotes the set of first n natural numbers. $\mathbb{Z}_n$ denotes the set $\{0, 1, \dots, n-1\}$. $G = (V, E)$ is a complete weighted graph without self-loops or parallel edges. V is the set of vertices and E is the set of edges.

We recall the definitions and well known results on round robin tournaments.

Definition 2.1 (*Single Round Robin Schedule*). Let $z \in \mathbb{N}$ be an even integer. A tournament schedule on z teams is called a single round robin schedule, if each team plays with every other team once.

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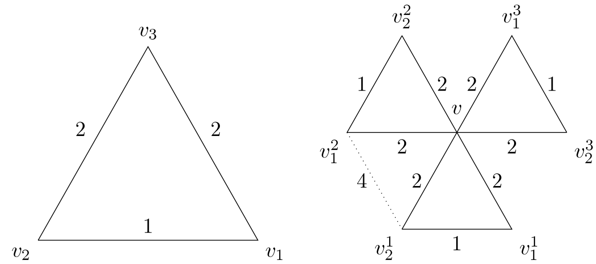
http://dx.doi.org/10.1016/j.orl.2016.07.011
0167-6377/



(a) The input instance G. (b) Corresponding W_3^3.

Fig. 1. (a): The input instance of TSP, the graph G, (b): Construction of $W_3^3[G]$.

We denote a schedule by a set of matches. Each match is represented by a 3-*tuple*, (a, b, r) where a plays at home against b in round r. The following theorem is folklore.

Theorem 2.2. *$\forall z \in \mathbb{N}$, there is a single round robin tournament of $2z$ teams.*

Definition 2.3 (*Double Round Robin Schedule*)**.** Let $n \in \mathbb{N}$ be an even integer. A tournament schedule on n teams is called a double round robin league schedule, if each team plays every other team twice; once at home and once away.

One can design a schedule of a double round robin tournament by repeating a single round robin tournament schedule with home-away reversed.

Unconstrained traveling tournament problem

We formulate the decision version of the Unconstrained Traveling Tournament Problem as follows:

- **Problem**: Traveling Tournament Problem
- **Instance**: An even integer n; an $n \times n$ matrix D; an integer K
- **Question**: Is there a double round robin league schedule of n teams located at venues represented by D, with total distance traveled by all the teams is at most K?

We represent an instance of this problem by $TTP(n, D, K)$. Without loss of generality we can assume D is a metric as any real life distance matrix will be a metric. To show that *UTTP* is *NP-hard*, we show a reduction from a variant of Traveling Salesman Problem; called (1, 2)-Traveling Salesman Problem ((1, 2)-TSP). Then decision version of (1, 2)-TSP is defined as follows

- **Problem**: (1, 2)- Traveling Salesman Problem
- **Instance**: A set $C = \{c_1, c_2, \ldots, c_n\}$ of n cities; distance $d(c_i, c_j) \in \{1, 2\}$ for all pair $c_i, c_j \in C, i \neq j$; an integer K.
- **Question**: Is there a cycle in C with cost at most K? In other terms, is there a permutation $\pi : [n] \to [n]$ such that
$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)}) \leq K?$$

It is easy to prove the *NP*-hardness of the (1, 2)-TSP using a reduction from the Hamiltonian Cycle problem. We refer the reader [6] for a detailed proof.

3. Construction of UTTP instance

We shall prove a reduction from (1, 2)-TSP to the unconstrained TTP. The reduction works in two parts. In the first part, we construct a modified instance of the metric TSP problem from the input instance. Next, we construct a UTTP instance based on the modified instance.

Modification of the TSP instance

Consider an input instance of (1, 2)-TSP on n cities, a complete weighted graph $G = (V, E)$ with $|V| = n$. $d(v_i, v_j) \in \{1, 2\}$ for all $i, j \in [n]$. Our objective is to construct a graph G' from G such that the optimal traveling salesman tour of G has cost K iff Traveling Salesman tour of G' is of cost nK. We construct such a graph using the well known windmill graph.

A windmill graph $K_n^{(l)}$ is the graph consisting of l copies of the complete graph K_n with a vertex in common across the copies. We construct an extended windmill graph which is a complete graph on $l(n-1)+1$ vertices, and assign weights based on whether the endpoints belong to same copy of or not.

Definition 3.1. Let $G = (V, E)$ be a complete weighted graph on n vertices $(n > 2)$. Let $d(i, j)$ denote the weight of each edge $(i, j) \in E$. The *extended windmill graph* $W_n^l[G]$ is a weighted complete graph on $l(n-1)+1$ vertices constructed from G in the following way

- Construct the windmill graph K_n^l from G taking l copies of V with v as the shared vertex. Let v_i^k denote the vertex v_i of the kth copy.
- For all $k \in [l]$, $d'(v_i^k, v) = d(v_i, v)$.
- For all $k \in [l]$, $d'(v_i^k, v_j^k) = d(v_i, v_j)$.
- $d'(v_i^{k_1}, v_j^{k_2}) = d(v_i, v) + d(v, v_j)$ where $k_1 \neq k_2$.

The vertex v is called the central vertex.

We note that the cost of each edge in $W_n^l[G]$ is at most 4.

In Fig. 1(b), we show the construction of $W_3^3[G]$ from the input instance of Fig. 1(a). We took three copies of G and contracted all the copies of v_3 to one vertex v. We showed only one edge between different copies to keep the image neat.

Proposition 3.2. *Let $G = (V, E)$ be a complete weighted graph on n vertices. Let $W_n^l[G]$ be the extended windmill graph. If triangle inequality holds in G, triangle inequality holds in $W_n^l[G]$.*

The proof of the triangle inequality for three arbitrary vertices $a, b, c \in W_n^l[G]$ follows by an elementary case distinction into the three cases where all the three vertices are in one copy of G, in two different copies of G, or in three different copies of G.

Lemma 3.3. *$W_n^l[G]$ has a traveling salesman tour of cost at most lK iff G has a traveling salesman tour of cost at most K.*

Proof. *If part*
Suppose, $\tau = \{v_1, v_2, \ldots, v_n, v_1\}$ be a traveling salesman tour of the graph G of cost K. Construct the corresponding $W_n^l[G]$ with v_1 as the central vertex. The tour $(v_1, v_2^1, \ldots, v_n^1, v_2^2, \ldots, v_n^2, \ldots, v_n^l, v_1)$ admits the cost lK.

Only if part
We need to prove that, if there is a traveling salesman tour of $W_n^l[G]$ of cost lK, G has a traveling salesman tour of cost at most K. We prove it by induction on l.