Operations Research Letters 44 (2016) 658–662

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Mean–variance portfolio selection with regime switching under shorting prohibition

Miao Zhang, Ping Chen *

Centre for Actuarial Studies, Department of Economics, University of Melbourne, Australia

A R T I C L E I N F O

Article history:
Received 23 March 2015
Received in revised form
7 July 2016
Accepted 7 July 2016
Available online 3 August 2016

Keywords:
Portfolio selection
Regime switching
Shorting prohibition
Mean–variance

A B S T R A C T

This paper investigates a mean–variance portfolio selection problem with regime switching under the constraint of short-selling being prohibited. By applying the dynamic programming approach, a system of Hamilton–Jacobi–Bellman (HJB) equations is constructed. Recognizing the features of the optimal wealth process, the optimal feedback control and verification theorem are obtained. The efficient portfolio and efficient frontier are explicitly derived through the Lagrange multiplier approach.

1. Introduction

Portfolio selection concerns how to choose the optimal proportions among a basket of securities in the financial market. In order to compare different allocation strategies, some criterion must be specified in the first place. Markowitz [4] proposed the first quantifiable measure of "mean–variance" by considering the trade-off between the expected return and variance of a portfolio in a single period model. Owing to mathematical difficulty, Markowitz's work has not been generalized to the multi-period case until several decades later when Li and Ng [2] firstly formulated and solved a multi-period mean–variance portfolio selection problem. In the continuous-time setting, Zhou and Li [8] obtained the efficient portfolio and efficient frontier in closed form by using embedding techniques.

In order to demonstrate the random nature of the financial market, there has been an increasing interest in regime switching models in which some of the finance parameters, such as stock appreciation rates and volatilities, are modulated by a continuous time Markov chain. See Sotomayor and Cadenillas [6], Zariphopoulou [7]. The regime-switching concept was originally applied to mean–variance portfolio selection by Zhou and Yin [9]. Chen et al. [1] extended their work by considering an uncontrollable liability process modelled by a Markov-modulated geometric Brownian motion.

Most of the existing literatures assume an ideal financial market where there is no limitation on short selling stocks. This is mainly due to mathematical tractability. Shorting stocks, however, is often restricted in the real world either by financial regulations or its trading cost. Thus it is more reasonable to incorporate shorting constraints into mean–variance modelling. One of these papers is by Li et al. [3] which prohibited short sale of stocks. Nevertheless, by assuming all the coefficients are deterministic, their model failed to capture the inherent uncertainty of the stock price evolution.

In this paper, we study a mean–variance portfolio selection problem with regime switching under no-shorting constraints. Under the stochastic linear-quadratic (LQ) control framework, the completion of square technique is not useful when control variables are constrained. To overcome this difficulty, Li et al. [3] constructed a continuous viscosity solution via two Riccati equations. However, the verification theorem is hard to prove when using their method. We address this theoretical problem by making use of the special features of the HJB equation and optimal wealth process in our model. In fact, the whole space $\{(t, x); t \geq 0, x \in R\}$ can be split into two areas, and within each area there is a smooth solution for the HJB equation whereas smoothness does not hold on the boundary. Fortunately, the optimal wealth process will always stay in one area when the interest rate is deterministic, which makes it possible to provide a verification theorem by merely applying the Itô formula.

The rest of the paper is organized as follows. Section 2 formulates a mean–variance portfolio selection problem. In Section 3, the Lagrange multiplier is introduced, and an unconstrained problem

* Corresponding author.
E-mail address: pche@unimelb.edu.au (P. Chen).

http://dx.doi.org/10.1016/j.orl.2016.07.008

is solved by the stochastic dynamic programming approach. Section 4 derives the efficient feedback portfolio and efficient frontier. Section 5 concludes this paper.

2. Problem formulation

Throughout the paper, we use the following notations. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space defined on which $W(t) = (W_1(t), \ldots, W_m(t))'$ is a m-dimensional standard Brownian motion. Let $\alpha(t)$ be a continuous time stationary Markov chain taking values in a finite state space $\mathcal{M} = \{1, 2, \ldots, l\}$ with generator matrix $Q = (q_{ij})_{l\times l}$. $W(t)$ and $\alpha(t)$ are independent of each other. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by $W(t)$ and $\alpha(t)$ augmented by null sets contained in $\mathcal{F}$. Define R_+^n as the set of all n-dimensional nonnegative vectors. The transpose of any matrix A is denoted by A'. The norm $\|\cdot\|$ is defined as $\|A\| = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$, where $A = (a_{ij})_{m\times n}$.

Suppose the financial market consists of n risky assets driven by Markov-modulated geometric Brownian motions. Let $p_i(t)$ denote the price of the ith risky asset which satisfies the following stochastic differential equation (SDE)

$$dp_i(t) = p_i(t)\left[b_i(t, \alpha(t))dt + \sum_{k=1}^{m} \sigma_{ik}(t, \alpha(t))dW_k(t)\right], \quad i = 1, 2, \ldots, n,$$

where $b_i(t, \alpha(t))$ and $\sigma_{ik}(t, \alpha(t))$ represent the appreciation rate and volatility of $p_i(t)$ respectively. There is one risk free asset whose price $p_0(t)$ is modelled by an ordinary differential equation (ODE)

$$dp_0(t) = r(t)p_0(t)dt,$$

where $r(t) > 0$ is the instantaneous interest rate independent of the market regimes $\alpha(t)$.

Consider an investor with an initial wealth x_0. Over a finite time horizon $T > 0$, the above $n + 1$ assets are traded continuously without any transaction cost. The investor's wealth process $x(t)$ may evolve as

$$\begin{cases} dx(t) = \left[r(t)x(t) + B(t, \alpha(t))'\mathbf{u}(t)\right]dt + \mathbf{u}(t)'\sigma(t, \alpha(t))dW(t), \\ x(0) = x_0, \quad \alpha(0) = i_0, \end{cases} \tag{2.1}$$

where i_0 is the initial market regime, and the n-dimensional vector process $\mathbf{u}(\cdot)$ is called a portfolio of the investor with each element representing the market value of each risky asset held by the investor. $B(t, \alpha(t))$ and $\sigma(t, \alpha(t))$ are defined as

$$B(t, \alpha(t)) = \Big(b_1(t, \alpha(t)) - r(t), \ldots, b_n(t, \alpha(t)) - r(t)\Big)', \quad \sigma(t, \alpha(t)) = \Big(\sigma_{ij}(t, \alpha(t))\Big)_{n\times m}.$$

We assume that the volatility matrix $\sigma(t, i)$ satisfies the non-degeneracy condition

$$\Sigma(t, i) := \sigma(t, i)\sigma(t, i)' \geq \delta I, \quad \forall t \in [0, T],\ i \in \mathcal{M},$$

where δ is some positive real number, and I is the $n \times n$ identity matrix.

Assumption 2.1. $r(\cdot), B(\cdot, \cdot), \Sigma(\cdot, \cdot)$ are Borel-measurable and uniformly bounded.

Assumption 2.2. $b_k(t, i) > r(t)$ for $k = 1, 2, \ldots, n$, $t \in [0, T]$, $i \in \mathcal{M}$.

Definition 2.1. A portfolio $\mathbf{u}(\cdot)$ is called admissible if $\mathbf{u}(\cdot)$ is a nonnegative square-integrable process. Let U denote the set of all admissible portfolios.

Proposition 2.1. *If $\mathbf{u}(\cdot) \in U$, then the corresponding wealth process $x(t)$ satisfies the integrability condition $E \max_{0\leq t\leq T} x(t)^2 < \infty$.*

Proof. If $\mathbf{u}(\cdot) \in U$, then SDE (2.1) admits a unique strong solution

$$x(t) = e^{\int_0^t r(s)ds}\left\{x_0 + \int_0^t e^{-\int_0^s r(v)dv} \times \left[B(s, \alpha(s))'\mathbf{u}(s)ds + \mathbf{u}(s)'\sigma(s, \alpha(s))dW(s)\right]\right\}.$$

Hence,

$$\begin{aligned} x^2(t) &\leq A\left\{\int_0^t B(s, \alpha(s))'\mathbf{u}(s)ds\right\}^2 + C\left\{\int_0^t \mathbf{u}(s)'\sigma(s, \alpha(s))dW(s)\right\}^2 + D \\ &\leq A\int_0^T \left[B(s, \alpha(s))'\mathbf{u}(s)\right]^2 ds + C\left\{\max_{0\leq t\leq T}\int_0^t \mathbf{u}(s)'\sigma(s, \alpha(s))dW(s)\right\}^2 + D, \end{aligned}$$

where A, C, D are suitable positive constants. Since $\mathbf{u}(\cdot)$ is admissible, the first integral above has a finite expectation. For the second one, applying Doob's Inequality yields

$$\begin{aligned} &E\left\{\max_{0\leq t\leq T}\int_0^t \mathbf{u}(s)'\sigma(s, \alpha(s))dW(s)\right\}^2 \\ &\leq 4E\left\{\int_0^T \mathbf{u}(s)'\sigma(s, \alpha(s))dW(s)\right\}^2 \\ &= 4E\int_0^T \|\mathbf{u}(s)'\sigma(s, \alpha(s))\|^2 ds < \infty. \end{aligned}$$

This completes the proof. □

The investor's objective is to find an admissible control $\mathbf{u}(\cdot)$ such that $\mathrm{Var}(x(T))$ is minimized provided $Ex(T) = z$ for any given expected return $z \in R_+$. This stochastic control problem can be formulated as follows

$$\begin{cases} \text{minimize} & \mathrm{Var}(x(T)) \\ \text{subject to} & Ex(T) = z,\ \mathbf{u}(\cdot) \in U. \end{cases} \tag{2.2}$$

When $z = x_0 e^{\int_0^T r(s)ds}$, problem (2.2) becomes trivial. In fact, the investor can achieve the expected return of $x_0 e^{\int_0^T r(s)ds}$ without uncertainties by only investing in the risk free asset. Under this scenario, the optimal portfolio is $\mathbf{u}(t) \equiv 0$. Therefore, in the following sections, we only investigate the nontrivial case when z is greater than $x_0 e^{\int_0^T r(s)ds}$. For any $z > x_0 e^{\int_0^T r(s)ds}$, the optimal portfolio is called an efficient portfolio and all the pairs $(z, \min \mathrm{Var}(x(T)))$ constitute the efficient frontier.

3. Decomposition of problem (2.2)

3.1. Lagrange multiplier

The mean–variance problem (2.2) is a convex minimization problem with a linear equality constraint, which could be well addressed by applying the technique of "Lagrange multiplier" in the following lemma.