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Solving minimax control problems via nonsmooth optimization

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ABSTRACT

We address minimax optimal control problems with linear dynamics. Under convexity assumptions, by using non-smooth optimization techniques, we derive a set of optimality conditions for the continuous-time case. We define an approximated discrete-time problem where analogous conditions hold. One of them allows us to design an easily implementable descent method. We analyze its convergence and we show some preliminary numerical results.

1. Introduction

We consider an optimal control problem with linear dynamics and fixed initial state where the goal is to minimize a cost functional which is the essential supremum, over the time interval, of a function depending on the time, the state and the control. Studied in the last decades by several authors [6,5,7,14,20,3,1,2,13] these problems differ from those with an accumulated cost criterion and arise naturally in many applications, as for instance, minimization of the maximum trajectory deviation from what is desired [11,12,15], or robust optimal control of uncertain systems [16,19].

Certainly, by adding an auxiliary variable the minimax control problem can be written as a classical control problem with state constraints, in this framework, some authors [5,14,20] obtained necessary conditions as Pontryagin Maximum Principle [18]. Nevertheless, in this case the adjoint state involves Radon measures, and therefore it is not easily implementable. It is also relevant the dynamic programming approach [6,3,1,2,13], which usually requires the discretization of the state space for computational implementations, leading to large scale problems. Since in this work we consider fixed initial state, our approach only requires time discretization, avoiding dimensionality drawbacks.

The main idea of this paper is to consider the minimax control problem as a non-smooth optimization problem in a suitable space. We follow [4] but now we focus on the discrete-time approximation in order to develop a numerical scheme. Under suitable assumptions we prove the existence of the cost functional directional derivatives and we derive a set of first order optimality conditions from which we design a descent method following the Armijo's rule [17].

2. Continuous-time problem

2.1. Main assumptions

We consider the dynamical system

$$\begin{cases} \dot{y}(t) = g(t, y(t), u(t)) \quad t \in [0, T], \\ y(0) = x \in \mathbb{R}^r, \end{cases} \tag{2.1}$$

where $g : [0, T] \times \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^r$ is a given function. In the notation above $y_u(t) \in \mathbb{R}^r$ denotes the state function and $u(t) \in \mathbb{R}^m$ the control. The optimal control problem consists in minimizing the functional $J : \mathcal{U} \mapsto \mathbb{R}$ defined as

$$J(u) := \text{ess sup} \{f(y_u(t)) : t \in [0, T]\}, \tag{2.2}$$

over the set of controls

$$\mathcal{U} = \{u : [0, T] \to U \subset \mathbb{R}^m : u(\cdot) \text{ measurable}\},$$

where U is a compact and convex set and $f : \mathbb{R}^r \to \mathbb{R}$ is given.

Let us now fix the standing assumptions that we will consider in this paper:

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(H1) g is linear and has the form:

$$g(t, y(t), u(t)) = A(t)y(t) + B(t)u(t) + C(t)$$

where $A : [0, T] \to \mathbb{R}^{r\times r}$, $B : [0, T] \to \mathbb{R}^{r\times m}$ and $C : [0, T] \to \mathbb{R}^r$ are Lipschitz continuous functions.

(H2) f is convex, Lipschitz continuous and continuously differentiable.

Remark 2.1. Under the above assumptions, for any $u \in \mathcal{U}$ the state equation (2.1) admits a unique solution y_u. Also, the function J is well defined, and the essential supremum is actually the maximum over $[0, T]$.

2.2. Optimality conditions

We consider the problem as a nonlinear optimization problem in $L^2[0, T]$. Note that if assumption (H1) holds and f is convex, then J is a convex function of u. If in addition f is a Lipschitz continuous function, then J is Lipschitz continuous on $\mathcal{U}$ endowed with the $L^2[0, T]$ norm. So the optimal control problem has solution, since we are minimizing a continuous and convex function over a convex, closed and bounded set of $L^2[0, T]$ (see [10]).

In order to obtain a necessary condition for u to be optimal, we would like to compute the gradient or, at least, a directional derivative of J for u along an admissible direction v. It is easy to see that because of the involved definition of J, it could not exist. Nevertheless, our assumptions on f guarantee the directional differentiability of J.

In the remainder we will note $J'(u, v)$ the directional derivative of the function J in u over the direction v, and by differentiable we understand Fréchet differentiable (see [9]). From now on, we suppose that assumptions (H1) and (H2) hold.

We denote by $T_{\mathcal{U}}(u)$ the tangent cone to $\mathcal{U}$ at u [9].

Proposition 2.1. *Under the above assumptions, the function J is directionally differentiable at any $u \in \mathcal{U}$ and the directional derivative in a direction $v \in T_{\mathcal{U}}(u)$ is given by*

$$J'(u; v) = \sup_{t\in C_u} \langle \nabla f(y_u(t)), z_v(t)\rangle , \tag{2.3}$$

where C_u is the set of critical times

$$C_u = \operatorname*{argmax}_{t\in[0,T]} f(y_u(t)), \tag{2.4}$$

and z_v solves the following differential equation

$$\begin{cases} \dot z(t) = & A(t)z(t) + B(t)v(t), \quad t \in [0, T] \\ z(0) = & 0. \end{cases} \tag{2.5}$$

Proof. By the convexity and differentiability of f, from [9] we know that J is directionally differentiable and for any direction v,

$$J'(u, v) = \sup_{t\in C_u} D_u f(y_u(t))v.$$

Let ϕ be the application $u \mapsto y_u$, then $D_u f(y_u(t))v = \langle \nabla f(y_u(t)), \phi'(u, v)\rangle$, and

$$\phi'(u, v) = z_v,$$

where z_v is the solution of (2.5). □

By classical continuous optimization, we know that a necessary optimality condition for u to be optimal is that every directional derivative is non-negative, for every direction in $T_{\mathcal{U}}(u)$ [8]. This condition turns to be also sufficient in the convex case. The last assertion is equivalent to

$$\inf_{v\in T_{\mathcal{U}}(u)} \sup_{t\in C_u} \langle \nabla f(y_u(t)), z_v(t)\rangle \ge 0. \tag{2.6}$$

Let us explicit the linear operator $v \mapsto z_v$. By the variation of constants formula, the solution of (2.5) is given by

$$z_v(t) = \int_0^t S_{ts}B(s)v(s)\mathrm{d}s,$$

where the matrix S_{ts} is a solution of the system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} S_{ts} = & A(t)S_{ts}, \quad t \in [s, T] \\ S_{ss} = & I. \end{cases} \tag{2.7}$$

Now, the directional derivative can be written as

$$J'(u; v) = \sup_{t\in C_u} \left\langle \nabla f(y_u(t)), \int_0^t S_{ts}B(s)v(s)\mathrm{d}s \right\rangle. \tag{2.8}$$

Defining for each $u \in \mathcal{U}$ and $t \in [0, T]$, the element of $L^2[0, T]$

$$q_{u,t}(s) := I_t(s)B^\top(s)S_{ts}^\top \nabla f(y_u(t)), \quad \forall s \in [0, T],$$

where $I_t(s)$ is equal to 1 if $s \le t$ and 0 otherwise, we can rewrite (2.8) as

$$J'(u; v) = \sup_{t\in C_u} \langle q_{u,t}, v\rangle , \tag{2.9}$$

where the last scalar product is in $L^2[0, T]$.

Theorem 2.1. *Let $u \in \mathcal{U}$, then u is optimal if and only if*

$$\inf_{v\in\mathcal{U}-u} \sup_{t\in C_u} \langle q_{u,t}, v\rangle = 0. \tag{2.10}$$

Proof. If u is a minimizer of J, then $\inf_{v\in T_{\mathcal{U}}(u)} J'(u; v) \ge 0$. By (2.9), the last assertion is equivalent to

$$\inf_{v\in T_{\mathcal{U}}(u)} \sup_{t\in C_u} \langle q_{u,t}, v\rangle \ge 0. \tag{2.11}$$

By (H1)–(H2), we can deduce that $q_{u,t}$ is bounded in $L^2[0, T]$ independently of t. Since $\mathcal{U}$ is convex, the infimum over $T_{\mathcal{U}}(u)$ in (2.11) coincides with the infimum over the set $\mathcal{U} - u$. Since $v = 0$ is an admissible direction, we have

$$\inf_{v\in\mathcal{U}-u} \sup_{t\in C_u} \langle q_{u,t}, v\rangle = 0.$$

Conversely, the sufficiency is straightforward from the convexity and directional differentiability of J (see [8]). □

Condition (2.10) involves the computation of the set of critical times associated to u. The dependence of this set with respect to the control can cause some troubles in the aim of designing an algorithm based on that condition. In order to avoid this complication, we propose other necessary conditions where the supremum is taken over the whole interval $[0, T]$.

In the remainder, we denote $\mathcal{U}_u := \mathcal{U} - u$ the set of admissible directions.

Theorem 2.2. *Condition (2.10) implies*

$$\inf_{v\in\mathcal{U}_u} \sup_{t\in[0,T]} \left\{f(y_u(t)) - J(u) + \langle q_{u,t}, v\rangle\right\} = 0. \tag{2.12}$$

Also, condition (2.12) implies

$$\inf_{v\in\mathcal{U}_u} \sup_{t\in[0,T]} \langle q_{u,t}, v\rangle = 0, \tag{2.13}$$

and for any $\rho > 0$,

$$\inf_{v\in\mathcal{U}_u} \sup_{t\in[0,T]} \left\{f(y_u(t)) - J(u) + \langle q_{u,t}, v\rangle\right\} + \frac{\rho}{2} \|v\|^2 = 0. \tag{2.14}$$