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Stochastic geometric optimization with joint probabilistic constraints

Jia Liu [a,b], Abdel Lisser [a,*], Zhiping Chen [b]

[a] Laboratoire de Recherche en Informatique (LRI), Université Paris Sud - XI, Bât. 650, 91405 Orsay Cedex, France

[b] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, PR China

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ABSTRACT

This paper discusses geometric programs with joint probabilistic constraints. When the stochastic parameters are normally distributed and independent of each other, we approximate the problem by using piecewise linear functions, and transform the approximation problem into a convex geometric program. We prove that this approximation method provides a lower bound. Then, we design a sequential convex optimization algorithm to find an upper bound. Finally, numerical tests are carried out on a stochastic shape optimization problem.

1. Introduction

Geometric programming is an important topic in operations research where the objective function and the constraints of the corresponding optimization problems have a special form. Geometric optimization has been studied for several decades, it was introduced by Duffin et al. in the late 1960s [4]. Applications of geometric programming can be found in several surveys papers, namely Ecker [6], Peterson [13] and Boyd et al. [1]. Numerous practical problems can be formulated as geometric programs, e.g., electrical circuit design problems [1], information theory [3], queue proportional scheduling in fading broadcast channels [16], mechanical engineering problems [18], economic and managerial problems [11], nonlinear network problems [10]. A geometric program can be formulated as

$$(GP)\quad \min_t g_0(t) \text{ s.t. } g_k(t) \leq 1, \quad k = 1, \ldots, K,\ t \in \mathbb{R}^M_{++} \tag{1}$$

with

$$g_k(t) = \sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}}, \quad k = 0, \ldots, K. \tag{2}$$

Usually, $c_i \prod_{j=1}^{M} t_j^{a_{ij}}$ is called a monomial where c_i need to be nonnegative and $g_k(t)$ is called a posynomial. We denote by Q the number of monomials in (1), and $\{I_k,\ k = 0, \ldots, K\}$ is the disjoint index sets of $\{1, \ldots, Q\}$.

Geometric programs are not convex with respect to t whilst they are convex with respect to $\{z : z_j = \log t_j,\ j = 1, \ldots, M\}$. Hence, interior point method can be efficiently used to solve geometric programs.

In real world applications, some of the coefficients in (1) may not be known precisely. Hence, stochastic geometric programming is used to model geometric problems with random parameters. For instance, individual probabilistic constraints have been used to control the uncertainty level of the constraints in (1) [5,15]:

$$P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1 - \epsilon_k, \quad k = 1, \ldots, K, \tag{3}$$

where ϵ_k is the tolerance probability for the kth constraint in (2).

In this paper, we consider the following joint probabilistic constrained stochastic geometric programs

$$(SGP)\quad \min_{t \in \mathbb{R}^M_{++}} E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \tag{4}$$

$$\text{s.t.}\quad P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1,\ k = 1, \ldots, K\right) \geq 1 - \epsilon. \tag{5}$$

Unlike [5,15], we require that the overall probability of meeting the K geometric constraints is above a certain probability level $1 - \epsilon$, where $\epsilon \in (0, 0.5]$.

* Corresponding author.
E-mail addresses: jia.liu@lri.fr (J. Liu), lisser@lri.fr (A. Lisser), zchen@xjtu.edu.cn (Z. Chen).

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Stochastic geometric programs with joint probabilistic constraints are a special case of joint probabilistic constrained problems. The latter were first considered by Miller and Wagner [12]. They showed that joint probabilistic constrained problems are equivalent to concave deterministic problems for uncorrelated random variables. When the right hand side is a multivariate normally distributed random vector, Prékopa [14] showed that the joint probabilistic constraint problems are convex. Iwata et al. [9] studied stochastic optimization problems with linear or nonlinear objective function and individual chance constraints. They used their approach in order to determine the optimum cutting conditions. The coefficients of the linear constraint and the right hand side are correlated and normally distributed. The authors derive a SOCP deterministic reformulation of the individual chance constraints. A convex approximation approach is proposed for linear programs with joint probabilistic constraints in [2]. When the coefficients $a_{ij}, i \in I_k, \forall k, j = 1, \dots, M$, are deterministic and $c_i, i \in I_k, \forall k$ are uncorrelated normally distributed random variables, Dupačová [5] and Rao [15] show that the probabilistic constraint (3) is equivalent to two deterministic constraints involving posynomials and common additional slack variables.

To the best of our knowledge, there are no in-depth research results on the stochastic geometric programs with joint probabilistic constraints. Hence, in this paper, we propose new approaches for solving problem (*SGP*) for pairwise independent normally distributed coefficients.

We first reuse the reformulation of individual probabilistic constraints in [5,15] to reformulate this problem as a biconvex problem. We use the standard variable transformation [1] in order to derive a convex reformulation. However, there is a quantile function of the standard normal distribution in the reformulation which is nonelementary. Therefore, we apply the piecewise linear approximation to $\log(\Phi^{-1}(e^{x_k})^2)$ rather than to $\Phi^{-1}(y_k)$ [2], which leads to a convex approximation. Moreover, we show that this approximation provides a lower bound, and it also converges to an equivalent reformulation of problem (4)–(5) when the number of segments goes to infinity.

We derive an upper bound by using the obtained biconvex problem with a new sequential convex approximation algorithm. Notice that the authors in [2] used the piecewise linear approximation to come up with an upper bound. Finally, numerical results with a stochastic shape optimization problem show the efficiency of the proposed approaches.

2. Stochastic geometric optimization under Gaussian distribution

We suppose that the coefficients $a_{ij}, i \in I_k, \forall k, j = 1, \dots, M$, are deterministic and the parameters $c_i, i \in I_k, \forall k$ are normally distributed and independent of each other, i.e., $c_i \sim N(E_{c_i}, \sigma_i^2)$ [5]. Moreover, we assume that $E_{c_i} \geq 0$. As c_i are independent of each other, constraint (5) is equivalent to

$$\prod_{k=1}^{K} P\left(\sum_{i\in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1-\epsilon. \tag{6}$$

By introducing auxiliary variables $y_k \in \mathbb{R}_+,\ k = 1, \dots, K$, (6) can be equivalently transformed into

$$P\left(\sum_{i\in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq y_k, \quad k = 1, \dots, K, \tag{7}$$

and

$$\prod_{k=1}^{K} y_k \geq 1-\epsilon, \quad 1 \geq y_k \geq 0,\ k = 1, \dots, K. \tag{8}$$

It is easy to see that for independent normally distributed $c_i \sim N(E_{c_i}, \sigma_i^2)$ [5], constraint (7) is equivalent to

$$\sum_{i\in I_k} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(y_k) \sqrt{\sum_{i\in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \leq 1,$$
$$k = 1, \dots, K. \tag{9}$$

Here, $\Phi^{-1}(y_k)$ is the quantile of the standard normal distribution $N(0, 1)$. However, biconvex inequalities (9) are still very hard to solve within an optimization problem [7].

2.1. Standard variable transformation

The standard variable transformation $r_j = \log(t_j), j = 1, \dots, M$, and $x_k = \log(y_k),\ k = 1, \dots, K$, applied to (8) and (9) leads to the following constraints:

$$\sum_{i\in I_k} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\}$$
$$+ \sqrt{\sum_{i\in I_k} \sigma_i^2 \exp\left\{\sum_{j=1}^{M} (2a_{ij} r_j + \log(\Phi^{-1}(e^{x_k})^2))\right\}} \leq 1,$$
$$k = 1, \dots, K, \tag{10}$$

$$\sum_{k=1}^{K} x_k \geq \log(1-\epsilon), \quad x_k \leq 0,\ k = 1, \dots, K. \tag{11}$$

$\Phi^{-1}(\cdot)$ is also called the probit function and can be expressed in terms of the inverse error function:

$$\Phi^{-1}(y_k) = \sqrt{2}\,\mathrm{erf}^{-1}(2y_k - 1), \quad y_k \in (0, 1).$$

The inverse error function is a nonelementary function which can be represented by the Maclaurin series:

$$\mathrm{erf}^{-1}(z) = \sum_{p=0}^{\infty} \frac{\lambda_p}{2p+1} \left(\frac{\sqrt{\pi}}{2} z\right)^{2p+1},$$

where $\lambda_0 = 1$ and $\lambda_p = \sum_{i=0}^{p-1} \frac{\lambda_i \lambda_{p-1-i}}{(i+1)(2i+1)} > 0,\ p = 1, 2, \dots$. Thus, we know that $\Phi^{-1}(y_k)$ is convex for $1 > y_k \geq 0.5$, and concave for $0 < y_k \leq 0.5$. Moreover, $\Phi^{-1}(y_k)$ is always monotonic increasing.

Under constraint (11), we have $0.5 \leq 1-\epsilon \leq y_k = e^{x_k} < 1$. Hence, we can only focus on the right part of $\Phi^{-1}(e^{x_k})$. This means the feasible set constrained by both (10) and (11) is convex. However, as $\Phi^{-1}(\cdot)$ is nonelementary, we still need to approximate it for practical use. Unlike the approximation method in [2], we approximate $\log(\Phi^{-1}(e^{x_k})^2)$ rather than $\Phi^{-1}(y_k)$ by a piecewise linear function.

2.2. Approximation of $\log(\Phi^{-1}(e^{x_k})^2)$

We choose S different linear functions:

$$F_s(x_k) = d_s x_k + b_s, \quad s = 1, \dots, S,$$

such that

$$F_s(x_k) \leq \log(\Phi^{-1}(e^{x_k})^2), \quad \forall x_k \in [\log(1-\epsilon), 0),$$
$$s = 1, \dots, S. \tag{12}$$

The expression $\log(\Phi^{-1}(e^{x_k})^2)$ is then approximated by a piecewise linear function

$$F(x_k) = \max_{s=1,\dots,S} F_s(x_k). \tag{13}$$

Constraints (12) and (13) guarantee that $F(x_k)$ provides a lower bound of $\log(\Phi^{-1}(e^{x_k})^2)$.