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Exact time-dependent solutions for the $M/D/1$ queue

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ABSTRACT

Time-dependent solutions to queuing models are very useful for evaluating the performance of real-world systems. However, because of their mathematical complexity, few available results exist. In this paper, we derive the time-dependent performance measures for an $M/D/1$ queue starting with a positive number of initial customers. Using the limiting property of an Erlang distribution, we obtain closed-form time-dependent formulas for the queue length and the waiting time. Furthermore, the time-dependent queue length probability in a busy period is derived.

1. Introduction

Queuing models have been widely applied to the performance analysis of real-world systems in a variety of fields such as communication systems and transportation systems. In practice, the time-dependent characteristics of queues are often more meaningful for system analysis; a good example of this is the 24-h traffic profile on the Severn Bridge studied by Griffiths et al. [9] that applied the transient queue length distribution of an $M/E_k/1$ queue.

Although the time-dependent behaviors of queuing models are very useful for performance evaluations, few available results exist because of their mathematical complexity. Early studies on the time-dependent queue length distribution of the $M/M/1$ queue were conducted by Luchak [12] and Saaty [13]. In [1,4,10,15,17], time-dependent behaviors of Markovian queues were studied. Ammar et al. [2] studied $M/M/1$ queue with balking and reneging. They presented the transient queue length distribution and the busy period density function.

Recently, Baek et al. [3], Griffiths et al. [8,9], and Leonenko [11] derived the transient solutions for $M/E_k/1$ queues. They obtained the time-dependent queue length probability in terms of the generalized modified Bessel function of the second type. For more detailed review on the time-dependent queuing systems, readers are referred to Schwarz et al. [14].

Studies on queues with deterministic service times are even more scarce. To the best of our knowledge, Garcia et al. [6] was the first study that presented the closed-form time-dependent queue length probability. They constructed a Markov chain for the number of customers of an $M/D/1/N$ queue at an arbitrary service completion epoch and obtained the transient queue length distribution. Since they derived the transient probability using matrix analytic method, the results are computationally efficient. The queue length formula for the $M/D/c$ queue was studied by Franx [5] using matrix analytic method.

In this paper, we analyze time-dependent behavior of the traditional $M/D/1$ queue. We obtain the main results using the well-known limiting property of the Erlang distribution and the results are summarized in Table 1.

To the best of authors knowledge, our proposed results have not appeared in the literature, and this is one of the contributions of this paper. Comparing to the results in Franx [5], our closed-form solutions are more complex. However, with our results, one can compute the exact system performance measures as the function of time, and we believe this is another contribution of our paper. In addition to the methods based on the limiting property, a direct probabilistic argument can be used to obtain the transient results. We also present the analytic approach based on probabilistic arguments in the online supplementary material (see Appendix A).

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Table 1
Time-dependent probability distributions of $M/D/1$ queue.

Notation	Description	Eq. no.
$Q_0^{(j)}(t),\ Q_n^{(j)}(t)$	Queue length probability at time t	Eqs. (3) and (8)
$W_q^{(j)}(t,x)$	Waiting time distribution at time t	Eqs. (9) and (14)
$\tilde{Q}_0^{(j)}(t),\ \tilde{Q}_n^{(j)}(t)$	Queue length probability at time t in a busy period	Eqs. (17) and (20)

2. Main results

This section presents our main results. We derive the time-dependent queue length probability, waiting time distribution, and queue length probability in a busy period of the $M/D/1$ queue in closed-form. To do so, we apply the limiting property of an Erlang random variable to the solutions of the $M/E_k/1$ queue in Griffiths et al. [7–9] and Baek et al. [3].

2.1. Time-dependent queue length probability

Consider an $M/D/1$ queuing system with infinite waiting room, arrival rate λ and deterministic service time $1/\mu$. We denote by N_0 the number of new initial customers at time $t = 0$. Let $N(t)$ as the number of customers at time t and we define the following probability:

$$Q_n^{(j)}(t) = Pr[N(t) = n | N_0 = j], \quad (n \ge 0,\ j \ge 0),\ (t > 0).$$

It is well-known that the Erlang random variable of order k with mean $1/\mu$ converges to the constant $1/\mu$ in distribution as k increases. Let $F_X^*(\theta; n, k\mu) = \left(\frac{k\mu}{\theta + k\mu}\right)^n$ as the Laplace transform (LT) of Erlang random variable with the order n and the mean $n/(k\mu)$. By taking $k \to \infty$, we have

$$\lim_{k\to\infty} F_X^*(\theta; k, k\mu) = \lim_{k\to\infty} \left(\frac{k\mu}{k\mu + \theta}\right)^k = \lim_{k\to\infty}\left(1 + \frac{\theta}{k\mu}\right)^{-k} = e^{-\frac{\theta}{\mu}}. \quad (1)$$

Note that Eq. (1) is the LT of the Dirac delta function. Thus, it is easy to obtain

$$\lim_{k\to\infty} F_X(t; k, k\mu) = U\left(t - \frac{1}{\mu}\right) \quad (2)$$

in which, $F_X(t; n, k\mu)$ is the (cumulative) distribution function (DF) of Erlang random variable with order n and mean $n/k\mu$ given as

$$F_X(t; n, k\mu) = \int_0^t \frac{(k\mu)^n \cdot u^{n-1} \cdot e^{-k\mu t}}{(n-1)!} du.$$

Using this limiting property, it is not difficult to guess that the transient solutions to the $M/D/1$ queue can be obtained by applying the limiting property to the transient solutions to the $M/E_k/1$ queue.

We define $P_{n,i}^{(j)}(t)$, $(n \ge 1,\ j \ge 0,\ i = 1, 2, \ldots, k)$ as the probability that the queue length is n, the remaining service phase is i at time t in the $M/E_k/1$ queue starting with j initial new customers. The order and the mean of the service time are assumed to be k and $1/\mu$, respectively. Let $P_0^{(j)}(t)$ be the probability that the queue length is 0 at time t. More details on $P_{n,i}^{(j)}(t)$ and $P_0^{(j)}(t)$ are presented in Griffiths et al. [7] and Luchak [12]. By the limiting property of the Erlang distribution, $Q_n^{(j)}(t)$ can be given as

$$Q_n^{(j)}(t) = \begin{cases} \lim_{k\to\infty} \sum_{i=1}^{k} P_{n,i}^{(j)}(t), & (t > 0,\ n \ge 1), \\ \lim_{k\to\infty} P_0^{(j)}(t), & (t > 0,\ n = 0). \end{cases}$$

We then have the following theorem.

Theorem 2.1. *Using the result in Luchak [12], we obtain*

$$Q_0^{(j)}(t) = \sum_{r=0}^{\infty} \frac{(\lambda t)^r e^{-\lambda t}}{r!} \cdot \left(1 - \frac{r}{\mu t}\right) \cdot U\left(t - \frac{r+j}{\mu}\right) = \sum_{r=0}^{\lfloor \mu t - j \rfloor} \frac{(\lambda t)^r e^{-\lambda t}}{r!} \cdot \left(1 - \frac{r}{\mu t}\right), \quad (t > 0), \quad (3)$$

where $\rho = \lambda/\mu$, $\lfloor x \rfloor$ is the greatest integer less than or equal to x and $U(t-a)$ is the unit-step function.

Proof. The proof is shown in section 1 in the online supplementary material (see Appendix A). □

Next, using the results in Griffiths et al. [7] and Luchak [12], we have

$$Q_n^{(j)}(t) = \lim_{k\to\infty} \left[B_1(t) + B_2(t) - B_3(t)\right], \quad (t > 0), \quad (4)$$

where

$$B_1(t) = \sum_{i=1}^{k}\sum_{r=0}^{\infty} \frac{(\lambda t)^{n-j+r} e^{-\lambda t}}{(n-j+r)!} \cdot \frac{(k\mu t)^{k(r+1)-i} e^{-k\mu t}}{[k(r+1)-i]!}, \quad (t > 0), \quad (5)$$

$$B_2(t) = \sum_{r=0}^{\infty} k\mu \int_0^t P_0^{(j)}(u) \cdot \frac{[\lambda(t-u)]^{n+r} e^{-\lambda(t-u)}}{(n+r)!} \cdot \frac{[k\mu(t-u)]^{k(r+1)-1} e^{-k\mu(t-u)}}{[k(r+1)-1]!} du, \quad (t > 0), \quad (6)$$

$$B_3(t) = \sum_{r=0}^{\infty} k\mu \int_0^t P_0^{(j)}(u) \cdot \frac{[\lambda(t-u)]^{n+r+1} e^{-\lambda(t-u)}}{(n+r+1)!} \cdot \frac{[k\mu(t-u)]^{k(r+1)-1} e^{-k\mu(t-u)}}{[k(r+1)-1]!} du, \quad (t > 0). \quad (7)$$

Theorem 2.2. *It follows from Eqs. (4)–(7) that*

$$Q_n^{(j)}(t) = \lim_{k\to\infty} \sum_{i=1}^{k} P_{n,i}^{(j)}(t) = \frac{(\lambda t)^{n+\lfloor \mu t \rfloor - j} \cdot e^{-\lambda t}}{(n + \lfloor \mu t \rfloor - j)!} + \sum_{r=0}^{\lfloor \mu t \rfloor - j - 1} \sum_{m=0}^{\lfloor \mu t \rfloor - j - r - 1} \frac{\left[\lambda\left(t - \frac{r+1}{\mu}\right)\right]^m e^{-\lambda\left(t - \frac{r+1}{\mu}\right)}}{m!} \times \left(1 - \frac{m}{\mu t - r - 1}\right)\left[1 - \frac{\lambda(r+1)}{\mu(n+r+1)}\right] \cdot \frac{\left[\frac{\lambda(r+1)}{\mu}\right]^{n+r} e^{-\frac{\lambda(r+1)}{\mu}}}{(n+r)!}, \quad (t > 0). \quad (8)$$

Proof. The proof is shown in section 2 in the online supplementary material (see Appendix A). □