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Uniqueness of equilibrium in a payment system with liquidation costs

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ABSTRACT

We study a financial network where forced liquidations of an illiquid asset have a negative impact on its price, thus reinforcing network contagion. We give conditions for uniqueness of the clearing asset price and liability payments. Our main result holds under mild and natural assumptions on the price impact function: monotonicity of the price impact function and strict monotonicity of the proceeds of liquidation in the liquidated quantity.

1. Introduction

We study a financial network, in which banks hold interbank liabilities, cash, and shares of an illiquid asset. The settlement of interbank liabilities may force banks to liquidate some shares of the illiquid asset. This has a negative impact on the price of the illiquid asset. Marking to market of banks' balance sheets reinforces network contagion: lower asset prices may force other banks to default on their interbank liability payments. This results in an entanglement of price mediated contagion and network mediated contagion.

We model the price impact by a given inverse demand function. In equilibrium, this leads to a clearing price and liability payments, given as solution of a fixed point equation. Existence of the fixed point follows by Tarski's fixed point theorem, as shown in [6]. Uniqueness has remained an open problem. In this paper, we prove uniqueness under some mild and natural technical assumptions.

A key assumption is that the cash proceeds from asset liquidations are strictly increasing in the number of shares liquidated. This assumption is economically reasonable, but is not satisfied in the influential paper [6] for all parameter choices for the exponential inverse demand function.

Our uniqueness result carries over to other interbank clearing mechanisms. For illustration we sketch the proof when there are different seniority classes of interbank liabilities.

We also provide an algorithm for computing the fixed point in our baseline model that terminates in at most m iterations, where m denotes the number of banks in the network. This algorithm is instructive as it corresponds to the actual cascade of bank defaults that leads to the equilibrium and thus has a clear economic interpretation.

Our paper is related to the strand of literature on interbank liability clearing where various mechanisms may reinforce network contagion, e.g, [9,1,2,4].

The remainder of the paper is as follows. In Section 2 we introduce the financial network. Section 3 contains our main result on the uniqueness of the clearing price and liability payments, which is proved in Section 4. In Section 5 we extend our uniqueness result to a financial network with different seniority classes. In Section 6 we provide an algorithm for computing the fixed point along with its economic interpretation.

2. Financial network

We consider the payment network model of [6] which extends the model of [7] to account for the price impact of the liquidation of external assets. The financial network consists of m interlinked financial institutions ("banks") $i \in [m] = \{1, \ldots, m\}$. Bank i holds $\gamma_i \geq 0$ units of a liquid asset (cash), and $y_i \geq 0$ units of an illiquid asset. Cash has value one. The illiquid asset has a positive

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fundamental value $P > 0$. The total illiquid asset holdings of the banks is denoted by $y_{tot} := \sum_{i\in[m]} y_i$.

Nominal interbank liabilities. Interbank liabilities are represented by a matrix of nominal liabilities (L_{ij}), where $L_{ij} \geq 0$ denotes the cash-amount that bank i owes bank j. The total nominal liabilities of bank i sum up to

$$L_i = \sum_{j\in[m]} L_{ij}.$$

Bank i in turn claims a total nominal cash amount of $\sum_{j\in[m]} L_{ji}$ from the other banks. The nominal balance sheet of bank i is then given by:

- Assets: $\gamma_i + \sum_{j\in[m]} L_{ji} + y_i P$,
- Liabilities: $L_i +$ nominal net worth.

The nominal cash balance is $\gamma_i + \sum_{j\in[m]} L_{ji} - L_i$.

Price impact of liquidations. If bank i's nominal cash balance is negative, then it has a liquidity shortfall and sells some of its shares of the illiquid asset. This has a negative price impact on the illiquid asset, which we model by an inverse demand function. We assume there is an outside market for the illiquid asset that can absorb the total illiquid asset holdings of the banks at a distressed price. It is beyond the scope of this paper to endogenize both the demand function for the illiquid asset and the financial network payments. Instead, we consider a given inverse demand function satisfying some mild technical assumptions and we analyze the interplay between the forced liquidations and the payment equilibrium in the network of interbank liabilities.

The inverse demand function $f(x)$ gives the equilibrium price for the illiquid asset when x units of the asset are sold. We assume that $f(x)$ satisfies

(i) $f(0) = P$;
(ii) $f(x)$ is continuous and non-increasing in $x \in [0, y_{tot}]$;
(iii) $xf(x)$ is increasing in $x \in [0, y_{tot}]$.

The first property states that in absence of liquidations the price is given exogenously by P. The second property states that the price is non-increasing with the excess supply x. The third property specifies that the cash proceeds from liquidations do increase with the liquidated quantity x.

If property (iii) does not hold, then the cash proceeds from liquidations do not increase or decrease with the number of liquidated shares, which means that the marginal price of the asset is zero or negative at those points where the function $xf(x)$ is non-increasing. Suppose there exists an interval $(x_0, x_1) \subset [0, y_{tot}]$ in which $xf(x)$ is decreasing. Since the inverse demand function is known by all banks, it is unreasonable to assume that these points could represent a price clearing equilibrium. Indeed, it would suffice to liquidated less, i.e., x_0 and earn more or just as much as liquidating quantity $x \in (x_0, x_1)$. Therefore, any reasonable specification of the inverse demand function $f(x)$ is such that on the interval $[0, y_{tot}]$, $xf(x)$ is non-decreasing. We assume the stronger property that $xf(x)$ is increasing, which is still reasonable when there are unconstrained agents in the economy who derive positive utility from holding the asset. More importantly, it turns out that property (iii) is necessary for the uniqueness of an equilibrium: [5] show that the model of [6] features multiple equilibria, while it satisfies (i) and (ii). As an example, the exponential function used in [6], $f(x) := Pe^{-\rho x}$ satisfies (iii) if and only if $\rho \leq \frac{1}{y_{tot}}$.

We denote by $P_{\min} = f(y_{tot})$ the price when the total illiquid asset holdings of the banks y_{tot} are sold. We then have

$$f(x) \geq P_{\min} > 0, \quad \text{for all } x \in [0, y_{tot}].$$

If the revenue from selling y_i units of the illiquid asset does not cover the negative cash-balance, then bank i defaults on its interbank liabilities. Interbank claims are of equal seniority, so that counterparty bank j will in turn receive a proportion

$$\Pi_{ij} = \begin{cases} L_{ij}/L_i & \text{if } L_i > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

of the cash-value of bank i's total assets. This means that the assets are distributed among the creditors according to the proportionality rule, see e.g. [7].

Negative price externalities resulting from liquidity shortages are intertwined with negative network externalities resulting from non-payment of liabilities. In the following we prove existence and uniqueness of the clearing equilibrium and provide a finite algorithm for the fixed point solution.

3. Existence and uniqueness of equilibrium

In equilibrium, the previous characterization of actual cash flows and price impact lead to a clearing price P^* and total liability vector $\mathbf{L}^* = (L_1^*, \ldots, L_m^*)$, which can be determined as a fixed point, $\Phi(P^*, \mathbf{L}^*) = (P^*, \mathbf{L}^*)$, of the non-linear map Φ on $[P_{\min}, P] \times [0, \mathbf{L}]$, with $\mathbf{L} = (L_1, \ldots, L_m)$, given by

$$\begin{cases} \Phi_0(p, \ell) = f\left(\sum_{i\in[m]} \frac{\left(L_i - \gamma_i - \sum_{j\in[m]} \ell_j \Pi_{ji}\right)^+}{p} \wedge y_i \right) \\ \Phi_i(p, \ell) = L_i \wedge \left(y_i \cdot p + \gamma_i + \sum_{j\in[m]} \ell_j \Pi_{ji} \right), \quad i \in [m]. \end{cases} \tag{2}$$

We have the following lemma.

Lemma 1. *The mapping Φ is monotone, continuous and bounded.*

Proof. First, note that $\Phi_0(p, \ell)$ is a non-decreasing continuous function of p and ℓ. Also for $i \in [m]$, we have that $\Phi_i(p, \ell)$ is a non-decreasing continuous function of p and ℓ, as it is the composition of the non-decreasing continuous maps $\ell \to yp + \gamma + \Pi^T \ell$ and $\ell \to \ell \wedge \mathbf{L}$. Last, note that, $\Phi(P_{\min}, \mathbf{0}) \geq (P_{\min}, \mathbf{0})$ and $\Phi(P, \mathbf{L}) \leq (P, \mathbf{L})$. This implies that the map Φ is bounded, which concludes the proof. □

As shown in [6,7], Lemma 1 and Tarski's fixed point theorem [10] implies the existence of a clearing price and total liability vector. However, uniqueness in the setup with price impact has remained an open problem. Our main result now solves this: uniqueness holds under very mild assumptions.

Theorem 2. *The mapping Φ has a unique fixed point if one of the following two conditions holds:*

(i) *all banks hold external assets, $y_i + \gamma_i > 0$ for all $i \in [m]$, or*
(ii) *the total of external assets is nonzero, $y_{tot} + \sum_{i\in[m]} \gamma_i > 0$, and the financial network is strongly connected, i.e., there is no subset $\mathcal{I} \subset [m]$ such that $\sum_{j\in\mathcal{I}} \Pi_{ij} = 1$ for each bank $i \in \mathcal{I}$.*

4. Proof of Theorem 2

The proof of Theorem 2 builds on the following lemma, which has immediate applications to other interbank liability clearing mechanisms.

Lemma 3. *Let f be the inverse demand function as above and let $\zeta : [P_{\min}, P] \to [0, y_{tot}]$ be a function satisfying*

1. *$\zeta(p)$ is continuous and non-increasing in $p \in [P_{\min}, P]$;*
2. *$p\zeta(p)$ is non-decreasing in $p \in [P_{\min}, P]$.*