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Semi-algebraic sets and equilibria of binary games

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ABSTRACT

Any nonempty, compact, semi-algebraic set in $[0, 1]^n$ is the projection of the set of mixed equilibria of a finite game with 2 actions per player on its first n coordinates. A similar result follows for sets of equilibrium payoffs. The proofs are constructive and elementary.

1. Introduction

In bimatrix games, the structure of the set of Nash equilibria is relatively well understood: this is a finite union of convex polytopes (Jansen [7]). Moreover, the possible sets of Nash equilibrium payoffs have been characterized by Lehrer et al. [8]: a subset E of $\mathbb{R}^2$ is the set of Nash equilibrium payoffs of a bimatrix game if and only if this is a finite union of rectangles with edges parallel to the axes; that is, of the form: $E = \cup_{1\leq i\leq m}[a_i, b_i] \times [c_i, d_i]$, where $m \in \mathbb{N}$ and $a_i, b_i, c_i, d_i \in \mathbb{R}$, with $a_i \leq b_i, c_i \leq d_i$.

For finite games with 3-players or more, the picture is much less clear. It is easily seen that the set of Nash equilibria or of Nash equilibrium payoffs is nonempty, compact and semi-algebraic; however, which semi-algebraic sets really arise as sets of Nash equilibria or of Nash equilibrium payoffs is not known. A few results have been obtained. For instance, Datta [6] showed that any real algebraic variety is isomorphic to the set of completely mixed Nash equilibria of a 3-player game, and also to the set of completely mixed equilibria of an N-player game in which each player has two strategies. More recently, Balkenborg and Vermeulen [1, Theorem 6.1] showed that any nonempty connected compact semi-algebraic set is homeomorphic to a connected component of the set of Nash equilibria of a finite game in which each player has only two strategies, all players have the same payoffs, and pure strategy payoffs are either 0 or 1. These results show that, modulo isomorphisms or homeomorphisms, and a focus on completely mixed equilibria or connected components of equilibria, all algebraic or nonempty compact semi-algebraic sets may be encoded as sets of Nash equilibria. We provide another result in this direction.

Since the set of Nash equilibria of an N-player finite game is a nonempty compact semi-algebraic subset of some $\mathbb{R}^k$, it follows from Tarski–Seidenberg's theorem that the projection of such a set on a subspace $\mathbb{R}^n$, $n < k$, satisfies the same properties. We prove a kind of converse of this fact: for any nonempty compact semi-algebraic set E, there exists a finite game with $N > n$ players, each having only two pure strategies, such that E is precisely the projection of the set of Nash equilibria of this game on its first n coordinates (those of the first n players). In this statement, we see a mixed strategy of an N-player game with two strategies per player as a vector $(x_1, \ldots, x_N)$ in $[0, 1]^N$; that is, we identify the strategy of the ith player with the probability x_i that it assigns to the first of its two strategies.

The above result implies a similar result on equilibrium payoffs, as opposed to equilibria: for any nonempty compact semi-algebraic set E in $\mathbb{R}^n$, there exists a finite game with $N > n$ players, each having only two pure strategies, such that E is precisely the set of Nash equilibrium payoffs of the first n players; that is, the projection of the set of Nash equilibrium payoffs on its first n coordinates (as will become clear, the "first n players" in our result on equilibrium payoffs have payoffs given by affine transformations of the strategies of the "first n players" in our result on equilibria). As discussed further in the next section, the result on equilibria has been obtained independently by Yehuda John Levy [9], who also obtained more general results on semi-algebraic functions and correspondences, but our techniques and precise results are different.

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Some differences with Datta [6] and Balkenborg and Vermeulen [1] should be stressed. First, in our result, there is no isomorphism or homeomorphism involved, but a projection on the first n players. Second, our results do not concern the set of completely mixed Nash equilibria, or a connected component of equilibria, but the whole set of equilibria. These are not related to algebraic varieties or to connected semi-algebraic sets, but to (nonempty compact) semi-algebraic sets, which need not be connected. Also note that there is a fundamental difference between the set of completely mixed Nash equilibria and the set of Nash equilibria: the first may be empty while the second cannot. This represents a conceptual difficulty: any construction needs to check at some point the nonemptiness of the input set. Third, our proofs are fully elementary. To be more precise, given a set and certificates of semi-algebraicity, closedness and nonemptiness, our construction does not use any results from real algebraic geometry. Starting with a game with n players with two strategies each and choosing their first strategies with probabilities $x_1, \ldots, x_n$, we show how to add additional players with two strategies such that in equilibrium, these additional players choose their first strategies with probabilities that are powers of the x_i, and how yet additional players then allow to build and combine any polynomial in $x_1, \ldots, x_n$ in order to obtain that the set of equilibrium strategies of the initial n players is a given (nonempty compact) semi-algebraic set in $[0, 1]^n$. A small modification of the game then allows to obtain a given (nonempty compact) semi-algebraic set in $\mathbb{R}^n$ as the set of Nash equilibrium payoffs of n players of this game. Note that, by contrast with the work of Yehuda John Levy, we provide a bound on the number of additional players in our construction, which is not far from being tight. Once again, this bound is obtained only by elementary arguments. Finally, when the semi-algebraic set is defined by polynomials with integer coefficients, we prove a more precise result ensuring that the constructed game has integer pure payoffs, at the cost of additional players. Once again the construction (given certificates) is elementary, but the bound on the number of players and on the size of the integer payoffs depends on precise results of algebraic geometry.

Note that if additional players are not restricted to have only two actions, Yehuda John Levy proved that only three such additional players are needed [9].

The remaining part of this note is organized as follows: we introduce some definitions and prove our main results in Section 2. Section 3 compares our work to that of Yehuda John Levy. Extensions are discussed in Section 4. Section 5 concludes.

2. Definition and main results

A subset of $\mathbb{R}^n$ is semi-algebraic if it may be obtained by finitely many unions and intersections of sets defined by a polynomial equality or strict inequality. By the finiteness theorem (see for example Proposition 5.1 in [3]), a closed semi-algebraic subset of $\mathbb{R}^n$ may also be described by finitely many unions and intersections of sets defined by a polynomial *weak* inequality. In particular, a compact subset E of $\mathbb{R}^n$ is semi-algebraic if and only if there exist positive integers A and B and polynomials in n variables P_{ab}, $1 \leq a \leq A$, $1 \leq b \leq B$ such that:

$$E = \bigcup_{a=1}^{A} \bigcap_{b=1}^{B} \{x \in \mathbb{R}^n,\ P_{ab}(x) \leq 0\}. \tag{1}$$

Let us say that a game is binary if each player has only two pure strategies. Note that our notion of binary games is weaker than the notion used by Balkenborg and Vermeulen [1]: they define a game to be binary if each player has two pure strategies, and if in addition, this is a common interest game (all players have the same payoff), with pure strategy payoffs always equal to 0 or 1.

Our first result is on equilibria. As before, we identify in its statement the mixed strategy of the ith player with the probability x_i that it assigns to the first of her two pure actions.

Proposition 1. *If E is a nonempty compact semi-algebraic subset of $[0, 1]^n$, then there exists an N-player binary game (with $N > n$) such that the projection of its set of Nash equilibria on its first n coordinates (those of the first n players) is equal to E.*

Our second result, a byproduct of our proof of Proposition 1, is on equilibrium payoffs.

Proposition 2. *If E is a nonempty compact semi-algebraic subset of $\mathbb{R}^n$, then there exists an N-player binary game (with $N > n$) such that the projection of its set of Nash equilibrium payoffs on its first n coordinates is equal to E.*

Essentially the same results have been independently obtained by Yehuda John Levy [9], but our techniques are different. Moreover, while the results of Yehuda John Levy are stronger in that Proposition 1 appears as a corollary of a more general result on semi-algebraic functions and correspondences, our proof is more elementary and we obtain a bound on the number of players needed: if E is described by (1), then both in Propositions 1 and 2,

$$N \leq 1 + AB + n(3 + 2\ln_2(d))$$

where d is such that each P_{ab} is of degree at most d in each variable.

The proof is constructive. It relies on appropriate gadget games, in the sense of algorithmic game theory. Before introducing these gadgets, we need to clarify our notation. We only consider binary games and we denote the two pure strategies of each player by Top and Bottom. It will be convenient to use the same piece of notation for the name of a player and its probability to play Top except that, to be able to distinguish between players and strategies, we use uppercase letters for players. Thus, the n basic players of the game bear the admittedly unusual names of players $X_1, X_2, \ldots, X_n$, and x_i is the probability that player X_i plays Top. The players we need to add are called players X_{ik}, Y_{ik}, S_{ab} or U. Player X_{ik} will be such that, in equilibrium, its probability x_{ik} of playing Top is equal to $(x_i)^{2^k}$. Since any positive integer is a sum of powers of 2, products of x_{ik} allow to obtain any power $(x_i)^q$ as the value in equilibrium of a multiaffine function of the probabilities used by the players of the game (where by multiaffine, we mean affine in each variable). Adding and multiplying such quantities allow to obtain the quantities $P_{ab}(x)$, where $x = (x_1, \ldots, x_n)$, as the value in equilibrium of the probability that an additional player S_{ab} plays Top. An additional gadget game then forces an additional player U to play Top when $(x_1, \ldots, x_n) \notin E$. Finally, the payoffs of the original players $X_1, \ldots, X_n$ are defined in such a way that, at any equilibrium in which U plays Top, $(x_1, \ldots, x_n) = (z_1, \ldots, z_n)$ where $z = (z_1, \ldots, z_n)$ is a fixed arbitrary element of the nonempty set E. Hence, at any equilibrium, if $x \notin E$ then $x = z \in E$ a contradiction. The converse, that is the fact that each $x \in E$ appears in an equilibrium, is an easy byproduct of the construction.

We define our binary games by giving the payoff of each player when she chooses Top or Bottom, as a function of the *mixed* strategy profiles of her opponents (more precisely, of their probabilities to play Top). These expressions will always be multiaffine in the probabilities of playing Top of the opponents (that is affine with respect to the probability to play Top of each opponent), ensuring that they correspond to payoffs in the mixed extension of a binary game. For instance, our first gadget game will have (at least) two players, say X_α and X_β, playing Top with probability x_α and x_β respectively, and with payoffs if they play Top or Bottom described by the following tables:

$$\text{Player } X_\alpha \ \begin{array}{c|c} T & x \\ \hline B & x_\beta \end{array} \qquad \text{Player } X_\beta \ \begin{array}{c|c} T & x_\alpha \\ \hline B & x \end{array} \tag{2}$$